Generic Identifiability of the DINA Model and Blessing of Latent Dependence

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Abstract

Cognitive Diagnostic Models are a powerful family of fine-grained discrete latent variable models in psychometrics. Within this family, the DINA model is a fundamental and parsimonious one that has received significant attention. Similar to other complex latent variable models, identifiability is an important issue for CDMs, including the DINA model. Gu and Xu (*Psychometrika*, 84(2):468-483, 2019) established the necessary and sufficient conditions for *strict* identifiability of the DINA model. Despite being the strongest possible notion of identifiability, strict identifiability may impose overly stringent requirements on designing the cognitive diagnostic tests. This work studies a slightly weaker yet very useful notion, *generic* identifiability, which means parameters are identifiable almost everywhere in the parameter space, excluding only a negligible subset of measure zero. We propose transparent generic identifiability conditions for the DINA model, relaxing existing conditions in nontrivial ways. Under generic identifiability, we also explicitly characterize the forms of the measure-zero sets where identifiability breaks down. In addition, we reveal an interesting *blessing-of-latent-dependence* phenomenon under DINA – that is, dependence between the latent attributes can restore identifiability under some otherwise unidentifiable **Q**-matrix designs. The blessing of latent dependence provides useful practical implications and reassurance for real-world designs of cognitive diagnostic assessments.

Keywords: Algebraic statistics; Blessing of dependence; Cognitive diagnostic model; DINA model; Generic identifiability; **Q**-matrix.

1 Introduction

Cognitive Diagnostic Models (CDMs), or Diagnostic Classification Models (Rupp and Templin, 2008b; von Davier and Lee, 2019), are a popular family of discrete latent variable

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models widely used in educational and psychological measurement. Based on subjects' multivariate responses to a set of items, one can use a CDM to infer the subjects' fine-grained latent attributes such as educational skills, personality traits, or mental disorders. The key structure in a CDM is the so-called $\mathbf{Q}$-matrix (Tatsuoka, 1983), which describes the item-attribute dependence relations. Various CDMs have been developed with different assumptions and modeling goals, among which the Deterministic Input Noisy output "And" gate model (DINA model; Junker and Sijtsma, 2001) is a very popular and basic one. The DINA model assumes a parsimonious conjunctive relationship among the latent attributes. It also serves as a submodel for more general CDMs such as the general diagnostic model (GDM; von Davier, 2008), the log linear CDM (LCDM; Henson et al., 2009), and the generalized DINA model (GDINA; de la Torre, 2011). Research into the DINA model has received enormous interest in the CDM literature (see, e.g. Rupp and Templin, 2008a; de la Torre, 2009; DeCarlo, 2011; Culpepper, 2015; Chen et al., 2018; Yamaguchi and Okada, 2020).

The rich modeling capacity of CDMs also comes with challenging identifiability issues. Generally, a model is said to be identifiable if parameters are uniquely recoverable from the observed data distribution. In CDMs, identifiability is the prerequisite for achieving reliable and valid diagnostic assessment of individuals. Recently, there have been emerging studies on the identifiability issues of CDMs. For general CDMs and their variants (which cover DINA as a submodel), several works have proposed identifiability conditions under different settings; see Xu (2017), Fang et al. (2019), Gu and Xu (2020), and Chen et al. (2020). A commonality among the above results for general CDMs is that, a $\mathbf{Q}$-matrix with K columns needs to contain at least two identity submatrices $\mathbf{I}_K$ for *strict identifiability*. However, such a condition is much stronger than needed for identifying the DINA model, which is a more parsimonious model yet exhibits rich combinatorial structures. In fact, Xu and Zhang (2016) first showed that when the $\mathbf{Q}$-matrix contains only one submatrix $\mathbf{I}_K$, the DINA model parameters can still be identifiable. Later, Gu and Xu (2019) further proposed sharp necessary and sufficient conditions for strict identifiability of DINA.

Concretely, strict identifiability means parameters are everywhere identifiable when coming from some parameter space $\mathcal{T}$. Despite being the strongest possible identifiability notion, strict identifiability may impose overly stringent conditions on the $\mathbf{Q}$-matrix design. A slightly weaker notion, *generic identifiability*, was introduced and studied by Allman et al. (2009) for various statistical latent structure models. Instead of requiring parameters to be everywhere identifiable in $\mathcal{T}$, generic identifiability allows there to be a measure-zero subset $\mathcal{N} \subseteq \mathcal{T}$ where identifiability breaks down. Such a notion can often suffice for real data analyses purposes (Allman et al., 2009) and hence is very useful in practice. For general CDMs and their variants, Gu and Xu (2020) and Chen et al. (2020) have proposed certain generic identifiability conditions. However, these generic identifiability conclusions are not applicable to the DINA model; rather, they are developed mostly for variants of CDMs that incorporate the *main effects* of latent attributes. It is an open problem whether generic identifiability of the DINA model can be established under more practical conditions on the $\mathbf{Q}$-matrix. Addressing this problem will positively inform the design of cognitive diagnostic assessments, and it may also inspire more efficient MCMC algorithms incorporating the more relaxed identifiability constraints (e.g., Kern and Culpepper, 2020).

To better understand generic identifiability of the DINA model, first recall Gu and Xu (2019)'s necessary and sufficient conditions for strict identifiability. These conditions can be summarized as three requirements on the $\mathbf{Q}$-matrix: *Completeness* (C), *Repeated measurement* (R), and *Distinctness* (D); see Section 2.2 for details. The necessity of each of the three conditions seemingly implies that they are equally important and that any violation of them would lead to similar outcomes of non-identifiability. However, it turns out that this is not the case. For example, the partial identifiability results in Gu and Xu (2020) imply that when Condition (C) is violated, certain parameters under DINA are always not identifiable, hence not generically identifiable. A natural question is, in violation of the minimal strict identifiability conditions, when will those unidentifiable parameters occupy most or all of the parameter space, and when will they only belong to a somewhat negligible subset of the parameter space (i.e., generically identifiable)? The above two scenarios have vastly different implications on the practice of $\mathbf{Q}$-matrix designs. It is therefore highly desirable to clarify the different nature of the minimal strict identifiability conditions, and to further establish weaker conditions for generic identifiability of the DINA model, if possible.

This work addresses the aforementioned questions and makes several contributions. First, we clarify that certain minimal conditions for strict identifiability of DINA are impossible to relax, in the sense that their violation will cause certain parameters to be nowhere identifiable in their parameter space. Second, we establish that certain other conditions can indeed be relaxed in nontrivial ways for generic identifiability to hold. The relaxed conditions only depend on the $\mathbf{Q}$-matrix structure and are easily checkable. Third, under generic identifiability, we explicitly characterize the forms of those measure-zero non-identifiable sets $\mathcal{N}$'s, and show that these sets correspond to certain independence statements about the latent attributes. This means the statistical dependence between latent attributes can help restore identifiability in some otherwise unidentifiable $\mathbf{Q}$-matrix settings. Therefore, the generic identifiability of DINA reveals an interesting *blessing-of-latent-dependence* phenomenon. This discovery has useful practical implications on designing real-world cognitive diagnostic assessments.

The remainder of this paper is organized as follows. Section 2 introduces the setup of the DINA model, reviews existing strict identifiability results, and motivates the study of generic identifiability. Section 3 presents necessary conditions and sufficient conditions for generic identifiability of the DINA model. Section 4 characterizes the forms of the measure-zero non-identifiable subsets under generic identifiability, and reveals the blessing-of-latent-dependence phenomenon. Section 5 provides some concluding remarks. The proofs of the theoretical results are all included in the Supplementary Material.

2 Model Setup and Strict Identifiability Results

2.1 DINA Model

We next introduce the notation of the DINA model proposed in Junker and Sijtsma (2001). Consider an educational assessment with J items designed to measure K binary latent attributes. For a random subject in the population, denote the observed response vector containing the responses to J items by $\mathbf{R} = (R_1, \dots, R_J)^\top$, where $R_j = 1$ or 0 denotes the correct or wrong response to the Jth item. Denote the random subject's latent attribute profile by $\mathbf{A} = (A_1, \dots, A_K)^\top$, where $A_k = 1$ or 0 represents the presence or absence of the kth latent attribute. Assume $\mathbf{A}$ follows a categorical distribution with *proportion parameters* $\boldsymbol{p} = (p_{\boldsymbol{\alpha}};\, \boldsymbol{\alpha} \in \{0,1\}^K)$; that is, $\mathbb{P}(\mathbf{A} = \boldsymbol{\alpha}) = p_{\boldsymbol{\alpha}}$ for any binary pattern $\boldsymbol{\alpha} \in \{0,1\}^K$. The proportion parameters $\boldsymbol{p}$ satisfy $\sum_{\boldsymbol{\alpha}\in\{0,1\}^K} p_{\boldsymbol{\alpha}} = 1$.

As for the distribution of the observables, assume a subject's responses $R_1, \dots, R_J$ are conditionally independent given his or her latent attribute profile $\mathbf{A}$. The key structure in specifying the conditional distribution of R_j's is the so-called $\mathbf{Q}$-matrix (Tatsuoka, 1983), which is an item-attribute matrix with binary entries. Specifically, $\mathbf{Q} = (q_{j,k})$ has rows indexed by the J items and columns by the K latent attributes, and $q_{j,k} = 1$ indicates item j *requires* or *measures* attribute k and $q_{j,k} = 0$ otherwise. Denote the J row vectors of $\mathbf{Q}$ by $\boldsymbol{q}_1, \dots, \boldsymbol{q}_J$. For any positive integer L, denote $[L] = \{1, 2, \dots, L\}$. For two vectors $\boldsymbol{\alpha} \in \{0,1\}^K$ and $\boldsymbol{q}_j$, we write $\boldsymbol{\alpha} \succeq \boldsymbol{q}_j$ if $\alpha_k \geq q_{j,k}$ for all $k \in [K]$; write $\boldsymbol{\alpha} \nsucceq \boldsymbol{q}_j$ otherwise. The DINA model adopts the conjunctive assumption of attributes, defining an ideal response $\xi_{j,\boldsymbol{\alpha}} = \mathbb{1}(\boldsymbol{\alpha} \succeq \boldsymbol{q}_j) = \prod_{k=1}^K \alpha_k^{q_{j,k}}$ for each item j and attribute pattern $\boldsymbol{\alpha}$. The $\xi_{j,\boldsymbol{\alpha}}$ is a binary indicator of whether pattern $\boldsymbol{\alpha}$ masters all the attributes required by the jth item, i.e., being *capable* of item j. The conditional distribution of R_j given $\mathbf{A}$ is then

$$\mathbb{P}(R_j = 1 \mid \mathbf{A} = \boldsymbol{\alpha}) = (1 - s_j)\xi_{j,\boldsymbol{\alpha}} + g_j(1 - \xi_{j,\boldsymbol{\alpha}}),$$
$$\mathbb{P}(R_j = 0 \mid \mathbf{A} = \boldsymbol{\alpha}) = 1 - \mathbb{P}(R_j = 1 \mid \mathbf{A} = \boldsymbol{\alpha}).$$

In the above expression, $s_j = \mathbb{P}(R_j = 1 \mid \xi_{j,\boldsymbol{\alpha}} = 1)$ denotes the *slipping* parameter, corresponding to the probability of slipping the correct response despite being capable of item j. And $g_j = \mathbb{P}(R_j = \mid \xi_{j,\boldsymbol{\alpha}} = 0)$ denotes the *guessing* parameter, corresponding to the probability of guessing the correct response despite being incapable of item j. Collect all of the slipping parameters by $\boldsymbol{s} = (s_j :\ j \in [J])$ and the guessing parameters by $\boldsymbol{g} = (g_j :\ j \in [J])$. We also call $\boldsymbol{s}$ and $\boldsymbol{g}$ the *item parameters*.

Under the aforementioned conditional independence assumption, the probability mass function of the observed vector $\mathbf{R}$ under the DINA model is

$$\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} \prod_{j=1}^{J} (1 - s_j)^{r_j \xi_{j,\boldsymbol{\alpha}}} g_j^{r_j (1-\xi_{j,\boldsymbol{\alpha}})} s_j^{(1-r_j)\xi_{j,\boldsymbol{\alpha}}} (1 - g_j)^{(1-r_j)(1-\xi_{j,\boldsymbol{\alpha}})}, \quad (1)$$

for any response pattern $\boldsymbol{r} \in \{0,1\}^J$. This completes the specification of the DINA model, with parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$.

2.2 Strict Identifiability of the DINA Model and Motivation for Generic Identifiability

In the statistics literature, strict identifiability of a model generally means that the parameters are everywhere identifiable in some parameter space $\mathcal{T}$. In the context of the DINA model, define the parameter space for $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ as

$$\mathcal{T} = \Big\{ (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) : \ 1 \geq 1 - s_j > g_j \geq 0 \text{ for all } j \in [J]; \quad (2)$$
$$p_{\boldsymbol{\alpha}} > 0 \text{ for all } \boldsymbol{\alpha} \in \{0,1\}^K, \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} = 1. \Big\}$$

First, the assumption $1 - s_j > g_j$ in (2) has been adopted for DINA model in many existing works (e.g., Culpepper, 2015; Xu and Zhang, 2016; Gu and Xu, 2019) to avoid trivial non-identifiability issues. Its interpretation is that capable subjects of an item always have a higher probability to give a correct response than incapable ones. Second, the assumption $p_{\boldsymbol{\alpha}} > 0$ for all $\boldsymbol{\alpha} \in \{0,1\}^K$ in (2) was also made in Xu and Zhang (2016) and Gu and Xu (2019) when deriving the C-R-D conditions, so it will make our new results in the same context as theirs. If, however, certain "attribute hierarchy" (Templin and Bradshaw, 2014) exists in that mastering attribute A_k is the prerequisite for mastering A_ℓ, then any latent attribute pattern $\boldsymbol{\alpha}$ with $\alpha_k = 0$ but $\alpha_\ell = 1$ will have population proportion zero $p_{\boldsymbol{\alpha}} = 0$. In such settings, the sharp conditions for strict identifiability will differ from the C-R-D conditions; see Gu and Xu (2022) for results in that context. In this work, we do not consider attribute hierarchies and focus on the same setting as Gu and Xu (2019) with the parameter space in (2), where C-R-D conditions are necessary and sufficient for strict identifiability. We define strict identifiability of the model parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ as follows.

Definition 1 (Strict Identifiability)**.** *The parameters* $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ *in the DINA model* (1) *associated with a* $\mathbf{Q}$*-matrix are said to be strictly identifiable, if for any set of valid parameters* $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \in \mathcal{T}$*, there exist no* $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ *such that*

$$\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \text{ for all } \boldsymbol{r} \in \{0,1\}^J.$$

We next summarize existing strict identifiability conditions for the DINA model. Xu and Zhang (2016) proposed a set of sufficient conditions and a set of necessary conditions for strict identifiability of DINA. Later, Gu and Xu (2019) bridged the gap between necessity and sufficiency, and further proposed minimal conditions on the $\mathbf{Q}$-matrix for strict identifiability. Specifically, Gu and Xu (2019) proved that the following three conditions (C), (R), and (D) are necessary and sufficient for strict identifiability of DINA.

(C) ***C****ompleteness.* A $\mathbf{Q}$-matrix with K columns contains an identity submatrix $\mathbf{I}_K$ after some row permutation. Namely, the $\mathbf{Q}$ can be row-permuted to take the form of

$$\mathbf{Q} = \begin{pmatrix} \mathbf{I}_K \\ \mathbf{Q}^* \end{pmatrix}. \tag{3}$$

(R) ***R****epeated-Measurement.* Each of the K attributes is required by at least three items. Namely, each column of $\mathbf{Q}$ contains at least three entries of "1"s.

(D) ***D****istinctness.* Assuming Condition (C) holds, after removing the identity submatrix $\mathbf{I}_K$ from $\mathbf{Q}$, the remaining submatrix of $\mathbf{Q}$ contains K distinct column vectors. (Namely, any two different columns of the submatrix $\mathbf{Q}^*$ in (3) are distinct.)

We will refer to Gu and Xu (2019)'s above three conditions as C-R-D conditions for short. We remark that in the CDM literature, the "completeness" of a $\mathbf{Q}$-matrix is not defined with respect to the $\mathbf{Q}$-matrix alone, but rather with respect to whether the $\mathbf{Q}$-matrix under a specific CDM can distinguish all the $|\{0,1\}^K| = 2^K$ possible attribute patterns (Chiu et al., 2009). Under the DINA model, however, it is indeed the case that a $\mathbf{Q}$-matrix is complete if and only if it contains an identity submatrix $\mathbf{I}_K$ (Chiu et al., 2009), so the name Completeness Condition (C) here under the DINA model should not cause confusion.

At first sight, the necessity of each of the three conditions (C), (R), and (D) seemingly implies that these conditions are of the same importance. One may conjecture that any violation of the C-R-D conditions would lead to the same unidentifiable outcome. However, it turns out that this is not the case. In fact, in violation of some of the C-R-D conditions, certain parameters are unidentifiable everywhere in their parameter space; while in violation of some others, those unidentifiable parameters only occupy a somewhat negligible measure-zero subset of the parameter space. The above two scenarios have vastly different implications on the practice of $\mathbf{Q}$-matrix designs. The latter scenario of $\mathcal{N}$ having measure zero is much more benign and often suffices for real data analyses purposes; this is generally called generic identifiability in the statistics literature (Allman et al., 2009). The next section will focus on developing generic identifiability results for the DINA model and delineate the different nature of the C-R-D conditions.

3 Generic Identifiability: Necessary Conditions and Sufficient Conditions

In this section, we consider whether the C-R-D conditions can be relaxed for generic identifiability of the DINA model; and if possible, how to relax them. We first define the concept of generic identifiability in the context of the DINA model.

Definition 2 (Generic Identifiability)**.** *The parameters* $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ *in the DINA model* (1) *associated with a* $\mathbf{Q}$*-matrix are said to be generically identifiable if the following holds. There exists a subset* $\mathcal{N}$ *of the parameter space* $\mathcal{T}$ *defined in* (2) *such that: (a)* $\mathcal{N}$ *has measure zero with respect to the Lebesgue measure on* $\mathcal{T}$*; and (b) for any set of parameters* $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \in \mathcal{T}\backslash\mathcal{N}$, *there exist no* $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ *such that*

$$\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \text{ for all } \boldsymbol{r} \in \{0,1\}^J.$$

The concept of generic identifiability was introduced and popularized by Allman et al. (2009) in the statistics literature. As the definition suggests, in a generically identifiable model, identifiability can only break down in a negligible subset $\mathcal{N}$ of the parameter space, so that parameter estimation and inference is still meaningful. Therefore, generic identifiability often suffices for real data analyses purposes and is very useful in practice.

3.1 Necessary Conditions for Generic Identifiability

Next, we first characterize two necessary conditions for generic identifiability. Namely, our results will delineate what conditions out of the C-R-D conditions *cannot* be relaxed for parameters to be identifiable almost everywhere in the parameter space.

Proposition 1 (Violation of the Completeness Condition (C))**.** *Under the DINA model, if the* $\mathbf{Q}$*-matrix violates Condition (C), then certain proportion parameters* $p_{\boldsymbol{\alpha}}$*'s are nowhere identifiable in their parameter space and generic identifiability fails to hold.*

In Proposition 1, "nowhere identifiable in their parameter space" means that regardless of where in the parameter space $\mathcal{T}$ these parameters come from, they are always not identifiable from the observed data distribution. This conclusion is a corollary from the $\boldsymbol{p}$-partial identifiability results in Gu and Xu (2020). We next elaborate more on Proposition 1. Recall the definition of the ideal response $\xi_{j,\boldsymbol{\alpha}} = \prod_{k=1}^K \alpha_k^{q_{j,k}}$, and define $\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} = (\xi_{1,\boldsymbol{\alpha}}, \xi_{2,\boldsymbol{\alpha}}, \ldots, \xi_{J,\boldsymbol{\alpha}})^\top$ as the ideal response vector for latent pattern $\boldsymbol{\alpha}$ across all the J items. For any two latent patterns $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^K$, if $\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} = \boldsymbol{\xi}_{:,\boldsymbol{\beta}}$, then the definition of the DINA model in (1) implies

$$\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \mathbf{A} = \boldsymbol{\alpha}, \boldsymbol{s}, \boldsymbol{g}) = \prod_{j=1}^J (1 - s_j)^{r_j \xi_{j,\boldsymbol{\alpha}}} g_j^{r_j(1-\xi_{j,\boldsymbol{\alpha}})} s_j^{(1-r_j)\xi_{j,\boldsymbol{\alpha}}} (1 - g_j)^{(1-r_j)(1-\xi_{j,\boldsymbol{\alpha}})}$$

$$= \prod_{j=1}^{J} (1-s_j)^{r_j \xi_{j,\boldsymbol{\beta}}} g_j^{r_j(1-\xi_{j,\boldsymbol{\beta}})} s_j^{(1-r_j)\xi_{j,\boldsymbol{\beta}}} (1-g_j)^{(1-r_j)(1-\xi_{j,\boldsymbol{\beta}})}$$
$$= \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \mathbf{A} = \boldsymbol{\beta}, \boldsymbol{s}, \boldsymbol{g}).$$

The above equality means the observed $\mathbf{R}$ depends on the latent $\mathbf{A}$ only through the ideal response vector $\boldsymbol{\xi}_{:,\mathbf{A}}$, so we can alternatively write $\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \mathbf{A} = \boldsymbol{\alpha}, \boldsymbol{s}, \boldsymbol{g})$ in an equivalent form as $\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g})$. With this observation, we can further rewrite the probability mass function of $\mathbf{R}$ as

$$\begin{aligned}
\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) &= \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} \cdot \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \mathbf{A} = \boldsymbol{\alpha}, \boldsymbol{s}, \boldsymbol{g}) \\
&= \sum_{\boldsymbol{\alpha} \in \{0,1\}^K} p_{\boldsymbol{\alpha}} \cdot \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g}) \\
&= \sum_{\boldsymbol{\alpha} \in \mathcal{R}} \Big(\sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} p_{\boldsymbol{\beta}} \Big) \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\beta}}, \boldsymbol{s}, \boldsymbol{g}), \qquad (4)
\end{aligned}$$

where $\mathcal{R} \subseteq \{0,1\}^K$ represents a set of the representative latent patterns, such that $\{\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{R}\}$ contains mutually distinct vectors and also covers all the possible ideal response vectors. Eq. (4) implies the distribution of $\mathbf{R}$ depends on the proportion $p_{\boldsymbol{\alpha}}$ only through the sum of proportions for those *equivalent* latent patterns, where "equivalence" is exactly in the sense of the equality of ideal response vectors $\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} = \boldsymbol{\xi}_{:,\boldsymbol{\beta}}$. Therefore, as long as there do exist equivalent patterns with $\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} = \boldsymbol{\xi}_{:,\boldsymbol{\beta}}$ for some $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, their separate proportion parameters $p_{\boldsymbol{\alpha}}$ and $p_{\boldsymbol{\beta}}$ are always not identifiable, and at best identifiable up to their sum. This shows a severe "nowhere identifiable" phenomenon for proportion parameters of equivalent attribute patterns. Importantly, there exist equivalent attribute patterns under DINA *if and only if* the $\mathbf{Q}$-matrix satisfies the Completeness Condition (C). Therefore, if a $\mathbf{Q}$-matrix does not contain a submatrix $\mathbf{I}_K$, certain proportion parameters $p_{\boldsymbol{\alpha}}$'s will always be unidentifiable regardless of what values these $p_{\boldsymbol{\alpha}}$'s take. This implies the failure of generic identifiability according to Definition 2. The following example gives a concrete illustration of the above phenomenon and Proposition 1.

Example 1. *Suppose the* $\mathbf{Q}$*-matrix takes the form*

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then the ideal response vectors are the same for the two latent attribute patterns $\boldsymbol{\alpha} = (00)$ *and* $\boldsymbol{\alpha}' = (01)$ *across all the items; namely* $\xi_{1,(00)} = \xi_{1,(01)} = 0$ *and* $\xi_{2,(00)} = \xi_{2,(01)} = 0$*, then* p_{00} *and* p_{01} *can be at best identifiable up to their sum* $p_{00} + p_{01}$*. This means the individual proportion parameter* p_{00} *(or* p_{01}*) is always not identifiable in a continuum within the interval* $(0,\ p_{00} + p_{01})$*, even if* $p_{00} + p_{01}$ *is already identified and known. Therefore, one should resort to the* $\boldsymbol{p}$*-partial identifiability notion proposed and studied by Gu and Xu (2020). Specifically, if* $\mathbf{Q}$ *violates Condition (C) just as in this example, Proposition 1 merely states it is impossible to have generic identifiability of all the individual parameters in the DINA model. In such scenarios, it may still be desirable and meaningful to study the identifiability of the item parameters* $(\boldsymbol{s}, \boldsymbol{g})$ *and the grouped proportion parameters (in this case, the* $\{p_{00} + p_{01},\ p_{10},\ p_{11}\}$*), i.e.,* $\boldsymbol{p}$*-partial identifiability.*

Proposition 1 indeed implies Condition (C) is necessary for generic identifiability of DINA. It is worth comparing this conclusion to the generic identifiability results in Gu and Xu (2020) and Chen et al. (2020) developed for CDMs with main effects. In fact, for a CDM that models the main effects of latent attributes (distinct from the conjunctive DINA, e.g., see Maris, 1999; von Davier, 2008; de la Torre, 2011), it is known that a $\mathbf{Q}$-matrix does not need to contain any identity submatrix $\mathbf{I}_K$ for generic identifiability to hold. Specifically, for main-effect-based CDMs, two $K \times K$ submatrices each with all diagonal elements being "1" and any off-diagonal element free to be either "1" or "0" plus some minor additional condition would deliver generic identifiability (Gu and Xu, 2020; Chen et al., 2020). In contrast, this is not the case for the DINA model as shown in Proposition 1 and Example 1.

Having understood Condition (C)'s role with respect to generic identifiability, we next continue to consider the Repeated-Measurement Condition (R). The following proposition shows that a "severe" violation of Condition (R) would also cause a severe outcome, implying the failure of generic identifiability.

Proposition 2 (Severe Violation of the Repeated-measurement Condition (R))**.** *Under the DINA model, suppose the* $\mathbf{Q}$*-matrix satisfies Condition (C) and severely violates Condition (R) in that some attribute is required by only one item (i.e.,* $\sum_{j=1}^{J} q_{j,k} = 1$ *for some* $k \in [K]$*). Then the item parameters* (s_j, g_j) *associated with this particular item and the proportion parameters* $\boldsymbol{p}$ *are nowhere identifiable in their parameter space, so generic identifiability fails to hold.*

Remark 1. Note that the earlier Proposition 1 already established that the violation of Condition (C) would lead to the failure of generic identifiability. Therefore, when considering Condition (R), to avoid Condition (C) from being the source of the failure of generic identifiability, we assume $\mathbf{Q}$ satisfies Condition (C) in Proposition 2.

Proposition 2 implies the condition that each attribute being required by at least two items (i.e., $\sum_{j=1}^{J} q_{j,k} \geq 2$ for all $k \in [K]$) is necessary for generic identifiability of DINA. The proof of Proposition 2 indicates that the non-identifiable outcome when $\sum_{j=1}^{J} q_{j,k} = 1$ for some $k \in [K]$ is similar in severity to the violation of Condition (C), in that even *local* identifiability fails to hold. This means even in an arbitrarily small local neighborhood of those parameters s_j, g_j, and $p_{\boldsymbol{\alpha}}$'s, there exist alternative parameters $\bar{s}_j$, $\bar{g}_j$, and $\bar{p}_{\boldsymbol{\alpha}}$'s that are indistinguishable from the true ones (i.e., giving rising to the same distribution of $\mathbf{R}$). Under such severe violations of the C-R-D conditions described in Propositions 1 and 2, estimation and inference of those unidentifiable parameters would not be meaningful.

3.2 Sufficient Conditions for Generic Identifiability

When the severe violations of C-R-D conditions in Propositions 1 and 2 do not occur, we next consider the setting where some latent attribute is required by only two items. We call this setting a "slight" violation of Condition (R), to distinguish it from the "severe" violation scenario in Proposition 2. It turns out that generic identifiability can hold in such settings, as shown in our next theorem. Let $\mathbf{1}_L$ denote a L-dimensional column vector whose entries all equal to one.

Theorem 1 (Slight Violation of the Repeated-measurement Condition (R))**.** *Consider the DINA model where* $\mathbf{Q}$ *satisfies Condition (C). Suppose some attribute is required by only two items (i.e.,* $\sum_{j=1}^{J} q_{j,k} = 2$ *for some* $k \in [K]$*). So the* $\mathbf{Q}$*-matrix can be written in the following form after some column/row permutation, where* $\mathbf{Q}^*$ *is a* $(J-2) \times (K-1)$ *submatrix and* $\boldsymbol{u}$ *is a* $(K-1) \times 1$ *vector.*

$$\mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u} \\ \hline \mathbf{0} & \mathbf{Q}^* \end{pmatrix}. \tag{5}$$

If the submatrix $\mathbf{Q}^*$ *satisfies the C-R-D conditions and* $\boldsymbol{u} \neq \mathbf{1}_{K-1}^{\top}$*, then the DINA model parameters* $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ *are generically identifiable.*

Remark 2. When we say a $J^* \times K^*$ submatrix $\mathbf{Q}^*$ of $\mathbf{Q}$ satisfies the C-R-D conditions, the Completeness Condition (C) refers to that $\mathbf{Q}^*$ contains an identity submatrix $\mathbf{I}_{K^*}$. In the context of Theorem 1, $K^* = K - 1$ and $\mathbf{Q}^*$ satisfies the C-R-D conditions, which means $\mathbf{Q}^*$ contains a submatrix $\mathbf{I}_{K-1}$. Generally, Condition (C) always requires a matrix to contain an identity submatrix that has the same number of columns as itself.

Remark 3. Note that although we write the $\mathbf{Q}$-matrix in Theorem 1 in a specific form where the first two items require the first attribute A_1, the identifiability conclusion indeed addresses the general case where an arbitrary attribute A_k is required by two arbitrary items. Given a general $\mathbf{Q}$-matrix in such a form, one can always rearrange the rows and columns in the $\mathbf{Q}$-matrix and check the identifiability conditions in Theorem 1.

We leave the mathematical characterization and statistical interpretation of the measure-zero non-identifiable set $\mathcal{N}$ to Section 4. Under the conditions in Theorem 1, the first attribute A_1 is only required by two items in $\mathbf{Q}$, yet all the parameters, including all individual proportions $p_{\boldsymbol{\alpha}}$'s, can be generically identifiable. Note that such a conclusion is substantially different from the $\boldsymbol{p}$-partial identifiability statements in Gu and Xu (2020), which states that when $\mathbf{Q}$ does not contain an $\mathbf{I}_K$, certain $p_{\boldsymbol{\alpha}}$'s are always not identifiable regardless of what values the true parameters take.

Theorem 1 addresses the case where one attribute is required by exactly two items. More generally, generic identifiability can hold even when multiple attributes are each required by two items. Our next theorem characterizes this conclusion rigorously. For several vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_L$ each of the same dimension K, define $\bigvee_{\ell=1}^{L} \boldsymbol{a}_\ell := (\max_{\ell=1}^{L} a_{\ell,1}, \ldots, \max_{\ell=1}^{L} a_{\ell,K})$ to be the elementwise maximum of these vectors.

Theorem 2. *Consider the DINA model where $\mathbf{Q}$ satisfies Condition (C). Suppose after some column and row permutation, the $\mathbf{Q}$-matrix takes the following form for some m, where the first $m+1$ latent attributes are each required by only two items.*

$$\mathbf{Q} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_1 \\ \hline \mathbf{0} & \mathbf{Q}^{(1)} \end{array}\right), \quad \mathbf{Q}^{(1)} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_2 \\ \hline \mathbf{0} & \mathbf{Q}^{(2)} \end{array}\right), \quad \cdots, \quad \mathbf{Q}^{(m)} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_{m+1} \\ \hline \mathbf{0} & \mathbf{Q}^{(m+1)} \end{array}\right). \tag{6}$$

For $\ell = 2, \ldots, m+1$, define $\widetilde{\boldsymbol{u}}^{(\ell)} = (\mathbf{0}, \boldsymbol{u}^{(\ell)})$ to be a $(K-1)$-dimensional vector, and denote $\widetilde{\boldsymbol{u}}^{(1)} = \boldsymbol{u}^{(1)}$, also a $(K-1)$-dimensional vector. Suppose $\bigvee_{\ell=1}^{m+1} \widetilde{\boldsymbol{u}}^{(\ell)} \neq \mathbf{1}_{K-1}^\top$. and the $(J-m-2) \times (K-m-1)$ matrix $\mathbf{Q}^{(m+1)}$ satisfies the C-R-D conditions. Then the DINA model parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ are generically identifiable.

Note that if all the $\boldsymbol{u}$-vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{m+1}$ in (6) are zero vectors, then the requirement $\bigvee_{\ell=1}^{m+1} \widetilde{\boldsymbol{u}}^{(\ell)} \neq \mathbf{1}_{K-1}^\top$ in Theorem 2 is satisfied and the first m attributes $A_1, \ldots, A_m$ are each measured by only two items. So Theorem 2 implies that generic identifiability can be achieved when Condition (R) is significantly weakened from requiring ≥ 3 items per attribute to ≥ 2 items per attribute.

Among the C-R-D conditions, thus far we have examined Condition (C) and Condition (R) regarding the requirements of generic identifiability. Next consider the Condition (D), which is a more complex combinatorial condition about the structure of a $\mathbf{Q}$-matrix. When Condition (D) is violated, after removing an identity submatrix $\mathbf{I}_K$, the remaining submatrix of $\mathbf{Q}$ contains at least a pair of identical column vectors. The next theorem shows that the DINA model can be generically identifiable in this scenario.

Theorem 3 (Violation of the Distinctness Condition (D))**.** *Consider the DINA model with a $\mathbf{Q}$-matrix satisfying Conditions (C) and (R), but violating Condition (D). Without loss of generality, suppose the $\mathbf{Q}$-matrix can be written in the following form after some column/row permutation,*

$$\mathbf{Q} = \left(\begin{array}{ccc} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \boldsymbol{v} & \boldsymbol{v} & \mathbf{Q}^* \end{array}\right); \text{ without loss of generality, one can write } \begin{pmatrix} \boldsymbol{v} & \boldsymbol{v} & \mathbf{Q}^* \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{Q}' \\ \mathbf{1} & \mathbf{1} & \mathbf{Q}'' \end{pmatrix}. \tag{7}$$

If the submatrix $\mathbf{Q}'$ satisfies the C-R-D conditions, and $\mathbf{Q}''$ contains some zero row vector, then the DINA model parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ are generically identifiable.

Remark 4. Note that if a $\mathbf{Q}$-matrix takes the form in (7) and satisfies Condition (R) that $\sum_{j=1}^{J} q_{j,k} \geq 3$ for each $k \in [K]$, then the binary vector $\boldsymbol{v}$ contains at least two entries of "1"s; i.e., the submatrix $\mathbf{Q}''$ has at least two rows. We would like to emphasize again that although the $\mathbf{Q}$-matrix in (7) takes a specific form where Condition (D) fails to hold for the first two attributes and the first two items are single-attribute items, Theorem 3 generally applies when Condition (D) fails for an arbitrary pair of attributes. Under the assumption of Theorem 3, one can always rearrange the columns and rows of $\mathbf{Q}$ to make it take the form of (7) and then check our identifiability conditions.

In summary, in this section we have shown that in violation of the Completeness Condition (C) or in severe violation of the Repeated-measurement Condition (R) (i.e., some attribute measured by only one item), certain parameters under DINA are always not identifiable and hence not generically identifiable. While a slight violation of Condition (R) (some attribute measured by two items) or a violation of the Distinctness Condition (D) can still yield a generically identifiable model under certain conditions. These conclusions delineate the different nature of the C-R-D conditions, the minimal conditions for strict identifiability, and hence characterize the fine borders between strict and generic identifiability. Furthermore, the new generic identifiability conditions provide relaxed and more practical requirements on the **Q**-matrix designs in practice.

4 Measure-zero Non-identifiable Sets and Blessing of Latent Dependence

This section will characterize the forms of the measure-zero non-identifiable subsets $\mathcal{N}$'s ($\mathcal{N} \subseteq \mathcal{T}$) under generic identifiability, and reveal the blessing-of-latent-dependence phenomenon. In general, it turns out that under the minimal conditions for generic identifiability of DINA, these null sets $\mathcal{N}$'s carry certain statistical independence interpretation of the latent attributes.

We fix some notation first. For a vector $\boldsymbol{z} = (z_1, \ldots, z_K)$ and some integers $1 \leq k < \ell \leq K$, denote $\boldsymbol{z}_{k:\ell} = (z_k, z_{k+1}, \ldots, z_\ell)$. For two random vectors (or variables) $\boldsymbol{x}$ and $\boldsymbol{y}$, write $\boldsymbol{x} \perp\!\!\!\perp \boldsymbol{y}$ if they are statistically independent, and $\boldsymbol{x} \not\perp\!\!\!\perp \boldsymbol{y}$ otherwise. For a pattern $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$, let $(1, \boldsymbol{\alpha}^*)$ and $(0, \boldsymbol{\alpha}^*)$ denote two K-dimensional binary patterns. The following theorem addresses the case of slight violation of the Repeated-measurement Condition (R).

Theorem 4. *Consider the generically identifiable setting in Theorem 1, where* $\mathbf{Q}$ *takes the following form, with the submatrix* $\mathbf{Q}^*$ *satisfying the C-R-D conditions and* $\boldsymbol{u} \neq \mathbf{1}_{K-1}^\top$.

$$\mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u} \\ \hline \mathbf{0} & \mathbf{Q}^* \end{pmatrix}.$$

(a) The measure-zero set $\mathcal{N}_{R,1} \subseteq \mathcal{T}$ where identifiability breaks down is characterized by

$$\begin{aligned}\mathcal{N}_{R,1} &= \{(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \in \mathcal{T} :\ \boldsymbol{p} \text{ satisfies } p_{(1,\boldsymbol{\alpha}_1^*)}p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)}p_{(1,\boldsymbol{\alpha}_2^*)} = 0 \ \ \forall \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}.\} \\ &= \{(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \in \mathcal{T} :\ \boldsymbol{p} \text{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u}).\}, \end{aligned} \tag{8}$$

where "$A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u}$" reads: latent attributes A_1 and $\mathbf{A}_{2:K}$ are conditionally independent given $\mathbf{A}_{2:K} \succeq \boldsymbol{u}$.

(b) In particular, parameters $\boldsymbol{s}_{3:J}$ and $\boldsymbol{g}_{3:J}$ are always identifiable; while the remaining parameters $\boldsymbol{s}_{1:2}$, $\boldsymbol{g}_{1:2}$, and $\boldsymbol{p}$ are identifiable as long as $\boldsymbol{p} \notin \mathcal{N}_{R,1}$.

Remark 5. When the form of a set $\mathcal{N} \subseteq \mathcal{T}$ depends only on the proportion parameters $\boldsymbol{p}$ but not on the item parameters $(\boldsymbol{s}, \boldsymbol{g})$, we will write $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \notin \mathcal{N}$ also simply as $\boldsymbol{p} \notin \mathcal{N}$ with a slight abuse of notation, just as in Theorem 4(b).

We will also call those non-identifiable measure-zero sets $\mathcal{N}$'s under generic identifiability by *null sets*. Note that the null set $\mathcal{N}_{R,1}$ in (8) is characterized by the zero-set of certain polynomials only involving the proportion parameters $\boldsymbol{p} = (p_{\boldsymbol{\alpha}} :\ \boldsymbol{\alpha} \in \{0,1\}^K)$. By the basic terminology in algebraic geometry, any simultaneous zero-set of several nonzero polynomials about a vector of parameters (such as those about $\boldsymbol{p}$ underlying $\mathcal{N}_{R,1}$) defines an *algebraic variety* (e.g., Allman et al., 2009). It is known that such an algebraic variety necessarily has measure zero with respect to the Lebesgue measure on the parameter space. In the proof of Theorem 4(a), we first establish that as long as the true proportions $\boldsymbol{p}$ satisfy $p_{(1,\boldsymbol{\alpha}_1^*)}p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)}p_{(1,\boldsymbol{\alpha}_2^*)} = 0$ for all $\boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^*$ with $\boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$, then the DINA model parameters are identifiable. Then based on such a defining characteristic of the null set $\mathcal{N}_{R,1}$ of proportions, we further derive the interpretation of conditional independence "$A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u}$" underlying $\mathcal{N}_{R,1}$ (see the proof of Theorem 4 for more details). In fact, interestingly, one will soon see the above observation holds more generally in that the null sets are often characterized by polynomial equations of the proportion parameters $p_{\boldsymbol{\alpha}}$'s, which carry the interpretation of latent independence.

The next theorem generalizes the blessing-of-latent-dependence phenomenon to the case where multiple attributes are each measured by only two items.

Theorem 5. *Consider the generically identifiable setting of Theorem 2, where* $\mathbf{Q}$ *takes the following form, with* $\mathbf{Q}^{(m+1)}$ *satisfying the C-R-D conditions and* $\bigvee_{\ell=1}^{m+1}(\mathbf{0},\ \boldsymbol{u}^{(\ell)}) \neq \mathbf{1}_{K-1}^{\top}$.

$$\mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_1 \\ \hline \mathbf{0} & \mathbf{Q}^{(1)} \end{pmatrix}, \quad \mathbf{Q}^{(1)} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_2 \\ \hline \mathbf{0} & \mathbf{Q}^{(2)} \end{pmatrix}, \quad \cdots, \quad \mathbf{Q}^{(m)} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_{m+1} \\ \hline \mathbf{0} & \mathbf{Q}^{(m+1)} \end{pmatrix}.$$

Define $p^{(\ell)}_{(z,\boldsymbol{\alpha}^*)} = \mathbb{P}(A_{\ell+1} = z,\ \mathbf{A}_{(\ell+2):K} = \boldsymbol{\alpha}^*)$ *for* $\ell = 0, \ldots, m$, *which characterizes the joint distribution of latent attributes* $A_{\ell+1}, \ldots, A_K$. *The measure-zero subset* $\mathcal{N}_R \subseteq \mathcal{T}$ *where identifiability may break down can be written as*

$$\mathcal{N}_R = \bigcup_{\ell=0}^{m} \mathcal{N}_\ell, \quad \textit{where}\ \ \mathcal{N}_\ell = \left\{ p^{(\ell)}_{(1,\boldsymbol{\alpha}_1^*)} p^{(\ell)}_{(0,\boldsymbol{\alpha}_2^*)} - p^{(\ell)}_{(0,\boldsymbol{\alpha}_1^*)} p^{(\ell)}_{(1,\boldsymbol{\alpha}_2^*)} = 0 \ \ \textit{for any}\ \ \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{w}^\ell \right\}$$

$$= \left\{ A_\ell \perp\!\!\!\perp \mathbf{A}_{(\ell+1):K} \;\middle|\; \mathbf{A}_{(\ell+1):K} \succeq \boldsymbol{w}^\ell \right\}, \quad \ell = 0, 1, \ldots, m;$$

here the notation $\boldsymbol{w}^\ell := \bigvee_{t=\ell+1}^{m+1}(\mathbf{0},\ \boldsymbol{u}^{(t)})$ *is also a* K*-dimensional binary vector.*

Remark 6. If $\boldsymbol{u} = \mathbf{0}$ is a zero vector in Theorem 1 and 4, then the conditional independence statement "$A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u}$" in the definition of the null set $\mathcal{N}_{R,1}$ becomes "$A_1 \perp\!\!\!\perp \mathbf{A}_{2:K}$", i.e., the marginal independence between A_1 and $\mathbf{A}_{2:K}$. This implies if some attribute A_k is required by only two items and both these items solely require A_k, the DINA parameters are identifiable if and only if $\mathbf{A}_k$ has any dependence on the remaining attributes. Similarly, if each $\boldsymbol{w}^\ell = \bigvee_{t=\ell+1}^{m+1}(\mathbf{0},\ \boldsymbol{u}^{(t)}) = \mathbf{0}$ is a zero vector in Theorem 2 and 5, then each set $\mathcal{N}_\ell$ characterizes the marginal independence $A_\ell \perp\!\!\!\perp \mathbf{A}_{(\ell+1):K}$.

Theorem 4 implies when Condition (R) is slightly violated in that some latent attribute A_k is required by only two items, then the model is identifiable as long as this attribute A_k has some statistical dependence on the remaining attributes. Theorem 5 further establishes a similar conclusion when multiple attributes are each required by only two items. Intuitively, we can understand this phenomenon in the following way. When some attribute is required by only two items, instead of three items as required for strict identifiability, then such lack of information in the **Q**-matrix part (i.e., in the *measurement part*) can be partly compensated for if there exists dependence between this attribute and remaining ones in the *latent part*. Therefore, Theorems 4 and 5 describes an interesting trade-off between the measurement part and the latent part of the DINA model, and reveals a nontrivial phenomenon of blessing of latent dependence on identifiability.

The next theorem characterizes the statistical implication of generic identifiability when the Distinctness Condition (D) is violated.

Theorem 6. *Consider the generically identifiable setting of Theorem 3, where* **Q** *takes the following form, with* $\mathbf{Q}'$ *satisfying the C-R-D conditions.*

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{Q}' \\ \mathbf{1} & \mathbf{1} & \mathbf{Q}'' \end{pmatrix}.$$

(a) *The measure-zero set* $\mathcal{N}_D = \mathcal{N}_{D,1} \cup \mathcal{N}_{D,2} \subseteq \mathcal{T}$ *where identifiability may break down can be written as follows,*

$$\mathcal{N}_{D,1} = \{\text{For all } \ \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}, \tag{9}$$

$$m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \cdot m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \cdot [m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)] = 0.\};$$

$$\text{and } \ \mathcal{N}_{D,2} = \{\text{For all } \ \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}, \tag{10}$$

$$m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \cdot m_3(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \cdot [m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_3(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)] = 0.\};$$

$$\text{where } \begin{cases} m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) := p_{(0,1,\boldsymbol{\alpha}_1^*)} p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,1,\boldsymbol{\alpha}_2^*)} p_{(1,0,\boldsymbol{\alpha}_1^*)}; \\ m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) := p_{(0,0,\boldsymbol{\alpha}_1^*)} p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)} p_{(1,0,\boldsymbol{\alpha}_1^*)}; \\ m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) := p_{(0,0,\boldsymbol{\alpha}_1^*)} p_{(0,1,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)} p_{(0,1,\boldsymbol{\alpha}_1^*)}. \end{cases} \tag{11}$$

Here each $m_i(\cdot, \cdot)$ can be viewed as a function which takes two arbitrary binary patterns in $\{0,1\}^{K-2}$ ($\boldsymbol{\alpha}_1^*$, $\boldsymbol{\alpha}_2^*$ or $\boldsymbol{\beta}_1^*$, $\boldsymbol{\beta}_2^*$) as input and outputs a value based on proportions $\boldsymbol{p}$.

(b) *In particular, all the item parameters except (g_1, g_2) and those proportion parameters $\{p_{(1,1,\boldsymbol{\alpha}^*)} : \forall \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\}$ are always identifiable. As for the remaining parameters, the g_1 and $\{p_{(1,0,\boldsymbol{\alpha}^*)} : \forall \boldsymbol{\alpha}^*\}$ are identifiable if $\boldsymbol{p} \notin \mathcal{N}_{D,1}$; the g_2 and $\{p_{(0,1,\boldsymbol{\alpha}^*)} : \forall \boldsymbol{\alpha}^*\}$ are identifiable if $\boldsymbol{p} \notin \mathcal{N}_{D,2}$; and $\{p_{(0,0,\boldsymbol{\alpha}^*)} : \forall \boldsymbol{\alpha}^*\}$ are identifiable if $\boldsymbol{p} \notin \mathcal{N}_{D,1} \cup \mathcal{N}_{D,2}$.*

The following proposition provides a statistical understanding about the null sets $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$ in Theorem 6 underlying generic identifiability.

Proposition 3. *Consider the measure-zero non-identifiable sets $\mathcal{N}_{D,1}$, $\mathcal{N}_{D,2} \subseteq \mathcal{T}$ defined in (9)-(10) in Theorem 6. The following statements hold,*

$$\begin{aligned} \mathcal{N}_{D,1} &\supseteq \{\boldsymbol{p} \textit{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_2 = 0)\} \supseteq \{\boldsymbol{p} \textit{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{3:K})\}; \\ \mathcal{N}_{D,2} &\supseteq \{\boldsymbol{p} \textit{ satisfies } (A_2 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_1 = 0)\} \supseteq \{\boldsymbol{p} \textit{ satisfies } (A_2 \perp\!\!\!\perp \mathbf{A}_{3:K})\}; \\ \mathcal{N}_{D,1} \cap \mathcal{N}_{D,2} &\supseteq \{\boldsymbol{p} \textit{ satisfies } (\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K} \mid \mathbf{A}_{1:2} \neq (1,1))\} \supseteq \{\boldsymbol{p} \textit{ satisfies } (\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K})\}. \end{aligned} \tag{12}$$

Namely, under the generic identifiability setting in Theorems 3 and 6, the measure-zero non-identifiable subsets $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$ cover (conditional) independence scenarios between the first two latent attributes for which Condition (D) fails, and the remaining latent attributes.

Remark 7. It is instrumental to compare the results in Theorem 3 and Proposition 3 to a toy example in Gu and Xu (2019). In particular, Example 2 in Gu and Xu (2019) constructed a $\mathbf{Q}$-matrix with $K = 2$ where Conditions (C) and (R) are both satisfied while Condition (D) is violated, in order to illustrate the necessity of Condition (D) for strict identifiability.

In fact, their **Q**-matrix can be viewed as just containing the first two columns of the **Q**-matrix in our Theorem 6. Gu and Xu (2019) showed that some parameters under DINA are always not identifiable under such a $J \times 2$ matrix, which would imply the failure of generic identifiability. In contrast, our Theorem 3 and Proposition 3 indicate that, if there are more than two attributes ($K > 2$) where there exists some statistical dependence between those two attributes for which Condition (D) fails and the remaining $K-2$ attributes, then generic identifiability can actually be easily restored. The new results thus deliver a highly nontrivial and reassuring insight into generic identifiability of the DINA model.

Our new results in Theorem 3 and Proposition 3 delineate the nature of Condition (D) as a "could-be-violated" condition for generic identifiability. Thus the violation of Condition (D) is somewhat similar in severity to the aforementioned slight violation of Condition (R). It is worth pointing out that the blessing of dependence happens regardless of the sign of the dependence. That is, as long as latent dependence exists (i.e., between A_1 and $\mathbf{A}_{2:K}$ in Theorem 4, or between $\mathbf{A}_{1:2}$ and $\mathbf{A}_{3:K}$ in Theorem 6), no matter whether it is positive or negative dependence, it will help restore identifiability anyway based on our technical proofs.

We would like to point out that the identifiability results in this work are of a very fine-grained nature obtained using quite nontrivial proof arguments. This can be seen from the explicit algebraic forms of the null sets and the nuanced identifiability conclusions about specific parameters in Theorems 4 and 6, for example. Importantly, such results *cannot* be obtained by invoking the Kruskal's Theorem (Kruskal, 1977) on the uniqueness of tensor decompositions, which is a popular and powerful tool for establishing identifiability of latent structure models (e.g., in Allman et al., 2009; Fang et al., 2019; Culpepper, 2019; Chen et al., 2020). In fact, identifiability conclusions obtained by invoking Kruskal's Theorem are usually of a rather "global" nature, where certain global rank conditions of the probability tensor delivers identifiability. In contrast, our proof focuses on marginal moments of the response vector, and investigate under which conditions any specific parameter (such as g_1, g_2 in Theorem 6) becomes identifiable from the polynomial equations given by the moments. Such a detailed analysis enables us to derive the explicit forms of those null sets $\mathcal{N}$'s under generic identifiability, and to obtain the statistical implication of the blessing of latent dependence.

5 Discussion

Although sharp conditions (the C-R-D conditions) previously exist for the strict identifiability of the DINA model, it was not clear how important each of them means to identifiability, and how one can relax them to obtain more practical generic identifiability conditions. In this work, we first delineated the fundamentally different nature of the C-R-D conditions, clarifying that certain violation (i.e., violation of Condition (C) and severe violation of Condition (R)) will cause some parameters to be always unidentifiable, and hence not generically identifiable. Then, we further proposed nontrivial relaxations of the C-R-D conditions that ensure generic identifiability. All of our new conditions only depend on the structure of the $\mathbf{Q}$-matrix and are easily checkable. Furthermore, under generic identifiability, we explicitly characterized the measure-zero subset of the parameter space where identifiability may break down, and reveal that these sets carry the interpretation of latent independence. In a nutshell, in slight violation of Condition (R) and Condition (D), as long as those latent attributes for which the conditions fail *depend on* the remaining latent attributes, identifiability can be restored. Therefore, the statistical dependence between latent attributes can help restore identifiability under some otherwise unidentifiable $\mathbf{Q}$-matrix designs.

One motivation for exploring as weak as possible identifiability conditions is that such results have useful implications on statistical estimation. For example, the Bayesian formulation and estimation of CDMs have recently gained a great surge of interest. A body of works including Culpepper (2015), Chen et al. (2018), Liu et al. (2020), Chen et al. (2021), Balamuta and Culpepper (2022) directly and cleverly incorporate identifiability constraints into designing Markov Chain Monte Carlo (MCMC) sampling algorithms. In addition, Kern and Culpepper (2020) used a special-case generic identifiability conclusion of DINA established in Gu and Xu (2021) to estimate a restricted four-parameter IRT model. Therefore, obtaining as weak as possible identifiability conditions for popular CDMs, such as DINA, may provide an impetus to design more efficient MCMC algorithms under less stringent conditions. Moreover, in educational assessment settings where the DINA model is believed to explain data well, obtaining more practical identifiability conditions can better inform the design of cognitive diagnostic tests.

Finally, the blessing of latent dependence intuitively reveals a trade-off between the measurement part and the latent part of the DINA model. The measurement part is characterized by the $\mathbf{Q}$-matrix, as $\mathbf{Q} = (q_{j,k})$ specifies how the observed R_j's depend on the latent A_k's. And the latent part is characterized by the proportion parameters $\boldsymbol{p}$, as $\boldsymbol{p} = (p_{\boldsymbol{\alpha}})$ encodes whether and how the latent attributes depend on each other. In violation of some of the C-R-D conditions on the $\mathbf{Q}$-matrix, there is a lack of information in the measurement part for identifiability to hold. In this case, the blessing of latent dependence reveals that, such lack of measurement information can be partly compensated for by the dependence information in the latent part, if any. This discovery not only is theoretically highly nontrivial, but also provides reassurance for applying the DINA model in practice. Indeed, in educational cognitive diagnosis, the multiple latent attributes are usually fine-grained skills falling within one ability domain, so they are highly likely to be dependent rather than independent of each other. In this regard, our theoretical results show that such latent dependence can be a blessing, rather than a concern.

Supplementary Material

The Supplementary Material contains the technical proofs of all the theoretical results.

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Supplementary Material

Before presenting the proofs of the identifiability results, we introduce a useful technical tool, the T-matrix of marginal response probabilities. This technical tool was proposed by Xu and Zhang (2016) and also used in Gu and Xu (2019) to study the identifiability of the DINA model. First, consider a general notation $\mathbf{\Theta} = (\theta_{j,\boldsymbol{\alpha}})_{J\times 2^K}$ collecting all of the item parameters under the DINA model. The $J \times 2^K$ matrix $\mathbf{\Theta}$ has rows indexed by the J items and rows by all of the $|\{0,1\}^K| = 2^K$ configurations of the binary latent attribute pattern, where the $(j,\boldsymbol{\alpha})$th entry $\theta_{j,\boldsymbol{\alpha}} = \mathbb{P}(R_j = 1 \mid \mathbf{A} = \boldsymbol{\alpha})$ denotes the probability of a positive response to the jth item given the latent attribute pattern $\boldsymbol{\alpha}$. Then under the conjunctive assumption of DINA, we can write $\theta_{j,\boldsymbol{\alpha}}$ as

$$\theta_{j,\boldsymbol{\alpha}} = \begin{cases} 1 - s_j, & \text{if } \xi_{j,\boldsymbol{\alpha}} = \prod_{k=1}^K \alpha_k^{q_{j,k}} = 1; \\ g_j, & \text{otherwise.} \end{cases}$$

Note that given a $\mathbf{Q}$-matrix, there is a one-to-one mapping between the matrix $\mathbf{\Theta}$ and the item parameters $(\boldsymbol{s}, \boldsymbol{g})$. We next define a $2^J \times 2^K$ matrix $T(\mathbf{\Theta})$ based on $\mathbf{\Theta}$. The rows of $T(\mathbf{\Theta})$ are indexed by the 2^J different response patterns $\boldsymbol{r} = (r_1, \dots, r_J)^\top \in \{0,1\}^J$, and columns by attribute patterns $\boldsymbol{\alpha} \in \{0,1\}^K$, while the $(\boldsymbol{r}, \boldsymbol{\alpha})$th entry of $T(\mathbf{\Theta})$, denoted by $T_{\boldsymbol{r},\boldsymbol{\alpha}}(\mathbf{\Theta})$, represents the marginal probability that subjects with latent pattern $\boldsymbol{\alpha}$ provide positive responses to the set of items $\{j : r_j = 1\}$, namely

$$T_{\boldsymbol{r},\boldsymbol{\alpha}}(\mathbf{\Theta}) = \mathbb{P}(\mathbf{R} \succeq \boldsymbol{r} \mid \mathbf{\Theta}, \boldsymbol{\alpha}) = \prod_{j=1}^J \theta_{j,\boldsymbol{\alpha}}^{r_j}.$$

We denote the $\boldsymbol{\alpha}$th column vector and the $\boldsymbol{r}$th row vector of the T-matrix by $T_{:,\boldsymbol{\alpha}}(\mathbf{\Theta})$ and $T_{\boldsymbol{r},:}(\mathbf{\Theta})$, respectively. The $\boldsymbol{r}$th element of the 2^J-dimensional vector $T(\mathbf{\Theta})\boldsymbol{p}$ is

$$T_{\boldsymbol{r},:}(\mathbf{\Theta})\boldsymbol{p} = \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} T_{\boldsymbol{r},\boldsymbol{\alpha}}(\mathbf{\Theta}) p_{\boldsymbol{\alpha}} = \mathbb{P}(\mathbf{R} \succeq \boldsymbol{r} \mid \mathbf{\Theta}, \boldsymbol{p}).$$

Based on the T-matrix, there is an equivalent definition of identifiability of $(\boldsymbol{\Theta}, \boldsymbol{p})$ (equivalently, identifiability of $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$). Further, the T-matrix has a nice property that will facilitate proving the identifiability results. We summarize them in the following lemma, whose proof can be found in Xu (2017).

Lemma 1. *Consider the DINA model defined in* (1)*.*

(a) The parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ are identifiable if and only if there does not exist $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}}) \neq (\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ such that

$$T(\boldsymbol{\Theta})\boldsymbol{p} = T(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}}.$$

(b) For any vector $\boldsymbol{\theta}^* = (\theta_1^*, \dots, \theta_J^*)^\top \in \mathbb{R}^J$, there exists an $2^J \times 2^J$ invertible matrix $D(\boldsymbol{\theta}^*)$ which depends only on $\boldsymbol{\theta}^*$ such that

$$T(\boldsymbol{\Theta} - \boldsymbol{\theta}^* \cdot \mathbf{1}_{2^K}^\top) = D(\boldsymbol{\theta}^*) \cdot T(\boldsymbol{\Theta}).$$

Lemma 1 (a) and (b) imply that for any vector $\boldsymbol{\theta}^* = (\theta_1^*, \dots, \theta_J^*)^\top$, there holds

$$T(\boldsymbol{\Theta} - \boldsymbol{\theta}^* \cdot \mathbf{1}_{2^K}^\top)\boldsymbol{p} = D(\boldsymbol{\theta}^*)T(\boldsymbol{\Theta})\boldsymbol{p} = D(\boldsymbol{\theta}^*)T(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}} = T(\bar{\boldsymbol{\Theta}} - \boldsymbol{\theta}^* \cdot \mathbf{1}_{2^K}^\top)\bar{\boldsymbol{p}} \tag{S.1}$$

The above equality will be frequently used throughout the proof of our identifiability results. In the following proofs, we sometimes will denote $\boldsymbol{c} := \mathbf{1}_J - \boldsymbol{s} = (1 - s_1, \dots, 1 - s_J)^\top$ for notational convenience. Using this notation, the DINA model parameters can be equivalently expressed as $(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{p})$.

S.1 Proof of Proposition 1

We rewrite Eq. (4) in the main text below,

$$\begin{aligned}
\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) &= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} p_{\boldsymbol{\alpha}} \cdot \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \mathbf{A} = \boldsymbol{\alpha}, \boldsymbol{s}, \boldsymbol{g}) \\
&= \sum_{\boldsymbol{\alpha}\in\{0,1\}^K} p_{\boldsymbol{\alpha}} \cdot \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g}) \\
&= \sum_{\boldsymbol{\alpha}\in\mathcal{R}} \Big(\sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} p_{\boldsymbol{\alpha}} \Big) \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g}),
\end{aligned}$$

where the notation $\mathcal{R} \subseteq \{0,1\}^K$ denotes a collection of representative latent attribute patterns, such that $\{\boldsymbol{\xi}_{:,\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{R}\}$ contains mutually distinct ideal response vectors and also covers all the possible ideal response vectors under the $\mathbf{Q}$-matrix. Because of (4), for any $\boldsymbol{\alpha} \in \mathcal{R}$, those patterns $\boldsymbol{\beta} \in \{0,1\}^K$ with $\boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}$ can be considered to be equivalent to $\boldsymbol{\alpha}$ under the DINA model with the considered $\mathbf{Q}$-matrix. For $\boldsymbol{\alpha} \in \mathcal{R}$, define the equivalence class of latent attribute patterns by

$$[\boldsymbol{\alpha}] := \{\boldsymbol{\beta} \in \{0,1\}^K : \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}\}.$$

We next show that if for some $\boldsymbol{\alpha} \in \{0,1\}^K$, the set $[\boldsymbol{\alpha}]$ contains multiple elements, say $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]$ with $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, then their corresponding proportion parameters $p_{\boldsymbol{\alpha}}$ and $p_{\boldsymbol{\alpha}'}$ will always be unidentifiable, no matter what values $p_{\boldsymbol{\alpha}}$ and $p_{\boldsymbol{\alpha}'}$ take. Specifically, if two sets of parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ and $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ satisfy that $\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ for all $\boldsymbol{r} \in \{0,1\}^J$ under a same $\mathbf{Q}$-matrix, then (4) gives

$$\sum_{\boldsymbol{\alpha}\in\mathcal{R}} \Big(\sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} p_{\boldsymbol{\alpha}} \Big) \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g}) = \sum_{\boldsymbol{\alpha}\in\mathcal{R}} \Big(\sum_{\substack{\boldsymbol{\beta}\in\{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} \bar{p}_{\boldsymbol{\alpha}} \Big) \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}});$$

and even if $(\boldsymbol{s}, \boldsymbol{g}) = (\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}})$, the identifiability equations $\mathbb{P}(\mathbf{R} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \mathbb{P}(\mathbf{R} \mid \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ only give the following,

$$\sum_{\boldsymbol{\alpha} \in \mathcal{R}} \Big(\sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} p_{\boldsymbol{\alpha}} - \sum_{\substack{\boldsymbol{\beta} \in \{0,1\}^K: \\ \boldsymbol{\xi}_{:,\boldsymbol{\beta}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}}} \bar{p}_{\boldsymbol{\alpha}} \Big) \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{\xi}_{:,\mathbf{A}} = \boldsymbol{\xi}_{:,\boldsymbol{\alpha}}, \boldsymbol{s}, \boldsymbol{g}) = 0, \quad \forall \boldsymbol{r} \in \{0,1\}^J.$$

From the above equations, one can not identify individual parameters $p_{\boldsymbol{\beta}}$ for those $\boldsymbol{\beta}$ belonging to a same equivalence class $[\boldsymbol{\alpha}]$. Next we will show that if $\mathbf{Q}$ violates the Completeness Condition (C), then some equivalence class $[\boldsymbol{\alpha}]$ will contain multiple elements, leading to the aforementioned non-identifiability consequence.

According to Gu and Xu (2020), the set of representative patterns $\mathcal{R}$ in (4) can be obtained using the row vectors of the $\mathbf{Q}$-matrix as follows,

$$\mathcal{R} = \left\{ \bigvee_{j \in S} \boldsymbol{q}_j : \ S \subseteq \{1, \ldots, J\} \text{ is an arbitrary subset of item indices} \right\}, \tag{S.2}$$

where $\bigvee_{j \in S} \boldsymbol{q}_j =: \boldsymbol{\alpha}$ denotes the elementwise maximum of the set of vectors $\{\boldsymbol{q}_j : \ j \in S\}$ and the kth entry of the resultant vector $\boldsymbol{\alpha}$ is $\alpha_k = \max_{j \in S}\{q_{j,k}\}$. So $\bigvee_{j \in S} \boldsymbol{q}_j$ is also a K-dimensional binary vector and hence $\mathcal{R} \succeq \{0,1\}^K$. In fact, $\mathcal{R} = \{0,1\}^K$ if and only if $\mathbf{Q}$ contains a submatrix $\mathbf{I}_K$ after some row permutation. To see this, consider if the row vectors of $\mathbf{Q}$ do not include a certain standard basis vector $\boldsymbol{e}_k$ (which has a "1" in the kth entry and "0" otherwise), then $\boldsymbol{e}_k$ does not belong to $\mathcal{R}$ defined in (S.2) because $\boldsymbol{e}_k$ cannot be written in the form of $\bigvee_{j \in S} \boldsymbol{q}_j$ for any subset $S \subseteq [J]$. Therefore, if $\mathbf{Q}$ violates the Completeness Condition (C), then $\mathcal{R}$ is a proper subset of $\{0,1\}^K$, which implies certain attribute patterns become equivalent under such a $\mathbf{Q}$-matrix. In summary, if a $\mathbf{Q}$-matrix does not contain a submatrix $\mathbf{I}_K$, certain proportion parameters $p_{\boldsymbol{\alpha}}$'s will always be unidentifiable regardless of the values of these $p_{\boldsymbol{\alpha}}$'s. This implies the failure of generic identifiability of the DINA model parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ according to Definition 2 and proves Proposition 1. $\square$

S.2 Proof of Proposition 2

The construction for non-identifiable parameters in this setting is the same as that in the proof of Theorem 1 in Xu and Zhang (2016). We next elaborate on this construction to make clear the failure of generic identifiability. Since $\mathbf{Q}$ satisfies Condition (C), we can write the form of $\mathbf{Q}$ as follows without loss of generality,

$$\mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{Q}^\star \end{pmatrix},$$

where the first attribute A_1 is required by only one item, the first item. Next construct two different sets of DINA model parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ and $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ which lead to the same distribution of $\mathbf{R}$. In particular, if setting $s_j = \bar{s}_j$ and $g_j = \bar{g}_j$ for all $j \geq 2$, then the identifiability equations $\mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) = \mathbb{P}(\mathbf{R} = \boldsymbol{r} \mid \bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ for all $\boldsymbol{r} \in \{0,1\}^J$ will exactly reduce to the following set of equations,

$$\forall \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}, \quad \begin{cases} p_{(0,\boldsymbol{\alpha}^*)} + p_{(1,\boldsymbol{\alpha}^*)} = \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,\boldsymbol{\alpha}^*)}; \\ g_1 p_{(0,\boldsymbol{\alpha}^*)} + (1-s_1) p_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_1 \bar{p}_{(0,\boldsymbol{\alpha}^*)} + (1-\bar{s}_1) \bar{p}_{(1,\boldsymbol{\alpha}^*)}. \end{cases}$$

The above system of equations involve $|\{\bar{g}_1, \bar{s}_1\} \cup \{\bar{p}_{\boldsymbol{\alpha}};\ \boldsymbol{\alpha} \in \{0,1\}^K\}| = 2^K + 2$ free unknown variables regarding $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$, while there are only 2^K equations, so there exist infinitely many different solutions to $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$. In particular, we can let $\bar{g}_1 = g_1$ and arbitrarily set $\bar{s}_1$ in a small neighborhood of s_1 with $\bar{s}_1 \neq s_1$. Then correspondingly solve for the proportion parameters $\bar{\boldsymbol{p}}$ as

$$\forall \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}, \quad \bar{p}_{(1,\boldsymbol{\alpha}^*)} = \frac{1-s_1}{1-\bar{s}_1} p_{(1,\boldsymbol{\alpha}^*)}, \quad \bar{p}_{(0,\boldsymbol{\alpha}^*)} = p_{(0,\boldsymbol{\alpha}^*)} + \left(1 - \frac{1-s_1}{1-\bar{s}_1}\right) p_{(1,\boldsymbol{\alpha}^*)}.$$

Since $\bar{s}_1$ can vary arbitrarily in the neighborhood of s_1 without changing the distribution of $\mathbf{R}$, we have shown that the parameter s_1 is always unidentifiable in the parameter space. The parameter g_1 can be similarly shown to be always unidentifiable. The fact that item parameters (s_1, g_1) are always unidentifiable whatever their values are indicates the failure of generic identifiability. This proves the conclusion of Proposition 2. $\square$

S.3 Proof of Theorem 1 and Theorem 4

Proof of Theorem 1. Below we rewrite the form of the $\mathbf{Q}$-matrix stated in the theorem,

$$\mathbf{Q} = \begin{pmatrix} 1 & \mathbf{0} \\ 1 & \boldsymbol{u} \\ \hline \mathbf{0} & \mathbf{Q}^\star \end{pmatrix}.$$

By Lemma 1, if parameters $(\boldsymbol{\Theta}, \boldsymbol{p})$ and $(\bar{\boldsymbol{\Theta}}, \bar{\boldsymbol{p}})$ give rise to the same distribution of the observed responses, then the following equality holds,

$$T_{\boldsymbol{r},:}(\boldsymbol{\Theta})\boldsymbol{p} = T_{\boldsymbol{r},:}(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}} \quad \text{for all} \quad \boldsymbol{r} \in \{0,1\}^J, \tag{S.3}$$

Note that the last $J-2$ rows of $\mathbf{Q}$ has the first column being an all-zero column, and has the other $K-1$ columns forming a sub-matrix $\mathbf{Q}^\star$ which satisfies the C-R-D conditions. Since the C-R-D conditions are sufficient for identifiability of DINA model parameters by Gu and Xu (2019), the last $J-2$ rows of the $\mathbf{Q}$-matrix implies a nice identifiability result for a subset of the model parameters $(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{p})$. We next elaborate on this observation.

For notational convenience, denote by $\mathbb{P}(\cdot)$ the probability under the true parameters $(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{p})$, and denote by $\overline{\mathbb{P}}(\cdot)$ the probability under the alternative parameters $(\bar{\boldsymbol{c}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$. For a $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$, let $(0, \boldsymbol{\alpha}^*)$, $(1, \boldsymbol{\alpha}^*) \in \{0,1\}^K$ denote two K-dimensional binary vectors. Since $\mathbf{Q}_{1,\,3:J}$ is an all-zero vector, it is always true that $\theta_{j,\,(1,\boldsymbol{\alpha}^*)} = \theta_{j,\,(0,\boldsymbol{\alpha}^*)}$ for $j \geq 3$ and $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$. Therefore, for any response pattern $\boldsymbol{r} = (r_1, r_2, \boldsymbol{r}^*) \in \{0,1\}^J$, Eq. (S.3) for $\boldsymbol{r}$ implies the following,

$$\sum_{(z,\boldsymbol{\alpha}^*)\in\{0,1\}^K} \prod_{j>2:\, r_j=1} \theta_{j,\,(0,\boldsymbol{\alpha}^*)} \cdot \mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, A_1 = z,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)$$

$$\sum_{\boldsymbol{\alpha}^*\in\{0,1\}^{K-1}} \prod_{j>2:\, r_j=1} \theta_{j,\,(0,\boldsymbol{\alpha}^*)} \cdot [\mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, A_1 = 1,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)$$

$$+ \mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, A_1 = 0,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)]$$

$$= \sum_{\boldsymbol{\alpha}^*\in\{0,1\}^{K-1}} \prod_{j>2:\, r_j=1} \bar{\theta}_{j,\,(0,\boldsymbol{\alpha}^*)} \cdot [\overline{\mathbb{P}}(R_1 \geq r_1,\, R_2 \geq r_2,\, A_1 = 1,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)$$

$$+ \overline{\mathbb{P}}(R_1 \geq r_1,\, R_2 \geq r_2,\, A_1 = 0,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)];$$

which can be further simplified to be

$$\begin{aligned} &\sum_{\boldsymbol{\alpha}^*\in\{0,1\}^{K-1}} \prod_{j>2:\, r_j=1} \theta_{j,\,(0,\boldsymbol{\alpha}^*)} \cdot \mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*) \\ =& \sum_{\boldsymbol{\alpha}^*\in\{0,1\}^{K-1}} \prod_{j>2:\, r_j=1} \bar{\theta}_{j,\,(0,\boldsymbol{\alpha}^*)} \cdot \overline{\mathbb{P}}(R_1 \geq r_1,\, R_2 \geq r_2,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*). \end{aligned} \tag{S.4}$$

Note that fixing an arbitrary (r_1, r_2) and varying $\boldsymbol{r}^* \in \{0,1\}^{J-1}$, the above systems of equations (S.4) can be viewed as surrogate identifiability equations $T(\boldsymbol{\Theta}^*)\boldsymbol{p}^* = T(\bar{\boldsymbol{\Theta}}^*)\bar{\boldsymbol{p}}^*$ for the last $J-2$ items in the test, where those $\theta_{j,(0,\boldsymbol{\alpha}^*)} =: \theta^*_{j,\boldsymbol{\alpha}^*}$ serve as surrogate item parameters $\boldsymbol{\Theta}^* = \{\theta^*_{j,\boldsymbol{\alpha}^*} : \ j = 3,\ldots,J;\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\}$; and those $\mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*) =: p^*_{\boldsymbol{\alpha}^*}$ serve as surrogate proportion parameters $\boldsymbol{p}^* = \{p^*_{\boldsymbol{\alpha}^*} : \ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\}$. An important observation is that the parameters $(\boldsymbol{\Theta}^*, \boldsymbol{p}^*)$ can be viewed as associated with the matrix $\mathbf{Q}^\star$ under a DINA model with $J-2$ items and $K-1$ latent attributes. Now that $\mathbf{Q}^\star$ satisfies the C-R-D conditions (which are sufficient for identifiability), we obtain the following "identifiability conclusions" for the parameters $(\boldsymbol{\Theta}^*, \boldsymbol{p}^*)$,

$$\begin{cases} \theta_{j,(0,\boldsymbol{\alpha}^*)} = \bar{\theta}_{j,(0,\boldsymbol{\alpha}^*)}; \\ \mathbb{P}(R_1 \geq r_1,\, R_2 \geq r_2,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*) = \overline{\mathbb{P}}(R_1 \geq r_1,\, R_2 \geq r_2,\, \mathbf{A}_{2:K} = \boldsymbol{\alpha}^*); \end{cases} \tag{S.5}$$

which hold for any $j \in \{3,\ldots,J\}$ and $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$. Recall that for any item $j \geq 3$, the parameter $\theta_{j,(0,\boldsymbol{\alpha}^*)}$ ranges over both item parameters c_j and g_j) when $\boldsymbol{\alpha}^*$ ranges in $\{0,1\}^{K-1}$, so the first part of (S.5) implies

$$c_j = \bar{c}_j, \quad g_j = \bar{g}_j, \quad \forall j \in \{3, \dots, J\}. \tag{S.6}$$

Recall the form of $\mathbf{Q}$ and the vector $\boldsymbol{u}$ stated in the theorem, for any $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$ and $\boldsymbol{\alpha}^* \succeq \boldsymbol{u}$ (i.e. vector $\boldsymbol{\alpha}$ is elementwisely greater than or equal to vector $\boldsymbol{u}$), the second part of (S.5) implies the following must hold,

$$(r_1, r_2) = \begin{cases} (0,0) \implies p_{(0,\boldsymbol{\alpha}^*)} + p_{(1,\boldsymbol{\alpha}^*)} = \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,\boldsymbol{\alpha}^*)}; \\ (1,0) \implies g_1 \cdot p_{(0,\boldsymbol{\alpha}^*)} + c_1 \cdot p_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_1 \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_1 \cdot \bar{p}_{(1,\boldsymbol{\alpha}^*)}; \\ (0,1) \implies g_2 \cdot p_{(0,\boldsymbol{\alpha}^*)} + c_2 \cdot p_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_2 \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_2 \cdot \bar{p}_{(1,\boldsymbol{\alpha}^*)}; \\ (1,1) \implies g_1 g_2 \cdot p_{(0,\boldsymbol{\alpha}^*)} + c_1 c_2 \cdot p_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_1 \bar{g}_2 \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_1 \bar{c}_2 \cdot \bar{p}_{(1,\boldsymbol{\alpha}^*)}. \end{cases} \tag{S.7}$$

First, we transform the system of equations (S.7) to obtain

$$\begin{cases} (g_1 - c_1) \cdot (g_2 - \bar{c}_2) \cdot p_{(0,\boldsymbol{\alpha}^*)} = (\bar{g}_1 - c_1) \cdot (\bar{g}_2 - \bar{c}_2) \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)}; \\ (g_2 - \bar{c}_2) \cdot p_{(0,\boldsymbol{\alpha}^*)} + (c_2 - \bar{c}_2) \cdot p_{(1,\boldsymbol{\alpha}^*)} = (\bar{g}_2 - \bar{c}_2) \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)}. \end{cases}$$

Note that the right hand sides of both the above equations are nonzero. So we can take the ratio of the two equations to obtain

$$f_1(\boldsymbol{\alpha}^*) := \frac{(g_1 - c_1) \cdot (g_2 - \bar{c}_2)}{(g_2 - \bar{c}_2) + (c_2 - \bar{c}_2) \cdot p_{(1,\boldsymbol{\alpha}^*)} / p_{(0,\boldsymbol{\alpha}^*)}} = \bar{g}_1 - c_1.$$

So for two arbitrary patterns $\boldsymbol{\alpha}_1^*$, $\boldsymbol{\alpha}_2^* \in \{0,1\}^{K-1}$ with $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$, our above deduction gives $f_1(\boldsymbol{\alpha}_1^*) = f_1(\boldsymbol{\alpha}_2^*) = \bar{g}_1 - c_1$. This equality of $f_1(\boldsymbol{\alpha}_1^*)$ and $f_1(\boldsymbol{\alpha}_2^*)$ implies

$$\begin{aligned} & (c_2 - \bar{c}_2) \cdot \frac{p_{(1,\boldsymbol{\alpha}_1^*)}}{p_{(0,\boldsymbol{\alpha}_1^*)}} = (c_2 - \bar{c}_2) \cdot \frac{p_{(1,\boldsymbol{\alpha}_2^*)}}{p_{(0,\boldsymbol{\alpha}_2^*)}}; \\ \implies \quad & (c_2 - \bar{c}_2) \cdot \left(\frac{p_{(1,\boldsymbol{\alpha}_1^*)}}{p_{(0,\boldsymbol{\alpha}_1^*)}} - \frac{p_{(1,\boldsymbol{\alpha}_2^*)}}{p_{(0,\boldsymbol{\alpha}_2^*)}} \right) = 0. \end{aligned} \tag{S.8}$$

The above equation indicates that as long as there exist one pair of patterns $\boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-1}$ with $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$ and $\boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^*$ such that

$$p_{(1,\boldsymbol{\alpha}_1^*)} p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)} p_{(1,\boldsymbol{\alpha}_2^*)} \neq 0, \tag{S.9}$$

then $p_{(1,\boldsymbol{\alpha}_1^*)}/p_{(0,\boldsymbol{\alpha}_1^*)} \neq p_{(1,\boldsymbol{\alpha}_2^*)}/p_{(0,\boldsymbol{\alpha}_2^*)}$ and we must have $c_2 = \bar{c}_2$ from (S.8). Under the assumption stated in Theorem 1 that $\boldsymbol{u} \neq \mathbf{1}_{K-1}$, there indeed exist such two distinct vectors $\boldsymbol{\alpha}_1^*$, $\boldsymbol{\alpha}_2^*$ satisfying $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$. Therefore, $c_2 = \bar{c}_2$ (i.e., c_2 is identifiable) as long as $\boldsymbol{p} \notin \mathcal{N}_{R,1}$, where the set $\mathcal{N}_{R,1}$ is defined in the statement of Theorem 4:

$$\mathcal{N}_{R,1} = \{\boldsymbol{p} \text{ satisfies } p_{(1,\boldsymbol{\alpha}_1^*)} p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)} p_{(1,\boldsymbol{\alpha}_2^*)} = 0 \text{ for any } \boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^* \text{ with } \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}\}.$$

Next, we transform the system of equations (S.7) in another way to obtain

$$\begin{cases} (c_1 - g_1) \cdot (c_2 - \bar{g}_2) \cdot p_{(1,\boldsymbol{\alpha}^*)} = (\bar{c}_1 - g_1) \cdot (\bar{c}_2 - \bar{g}_2) \cdot \bar{p}_{(1,\boldsymbol{\alpha}^*)}; \\ (g_2 - \bar{g}_2) \cdot p_{(0,\boldsymbol{\alpha}^*)} + (c_2 - \bar{g}_2) \cdot p_{(1,\boldsymbol{\alpha}^*)} = (\bar{c}_2 - \bar{g}_2) \cdot \bar{p}_{(1,\boldsymbol{\alpha}^*)}. \end{cases}$$

The ratio of the above two equations gives

$$f_2(\boldsymbol{\alpha}^*) := \frac{(c_1 - g_1) \cdot (c_2 - \bar{g}_2)}{(g_2 - \bar{g}_2) \cdot p_{(0,\boldsymbol{\alpha}^*)}/p_{(1,\boldsymbol{\alpha}^*)} + (c_2 - \bar{g}_2)} = \bar{c}_1 - g_1.$$

Again we have $f_2(\boldsymbol{\alpha}_1^*) = f_2(\boldsymbol{\alpha}_2^*)$ for any $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$ with $\boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^*$. Such an equality implies

$$(g_2 - \bar{g}_2) \cdot \frac{p_{(0,\boldsymbol{\alpha}_1^*)}}{p_{(1,\boldsymbol{\alpha}_1^*)}} = (g_2 - \bar{g}_2) \cdot \frac{p_{(0,\boldsymbol{\alpha}_2^*)}}{p_{(1,\boldsymbol{\alpha}_2^*)}}, \quad \Longrightarrow \quad (g_2 - \bar{g}_2) \cdot \left(\frac{p_{(0,\boldsymbol{\alpha}_1^*)}}{p_{(1,\boldsymbol{\alpha}_1^*)}} - \frac{p_{(0,\boldsymbol{\alpha}_2^*)}}{p_{(1,\boldsymbol{\alpha}_2^*)}} \right) = 0.$$

Therefore, as long as $\boldsymbol{p} \notin \mathcal{N}_{R,1}$, we also have $g_2 = \bar{g}_2$ and g_2 is identifiable.

Now note that the systems of equations (S.7) are symmetric about (c_1, g_1) and (c_2, g_2). Since we have already obtained $c_2 = \bar{c}_2$ and $g_2 = \bar{g}_2$ if $\boldsymbol{p} \notin \mathcal{N}_{R,1}$, we also have $c_1 = \bar{c}_1$ and $g_1 = \bar{g}_1$ if $\boldsymbol{p} \notin \mathcal{N}_{R,1}$ following the same argument. Thus far we have already established $\boldsymbol{c} = \bar{\boldsymbol{c}}$ and $\boldsymbol{g} = \bar{\boldsymbol{g}}$, i.e., have shown the identifiability of all the item parameters in $\boldsymbol{\Theta}$.

Since the item parameters $(\boldsymbol{c}, \boldsymbol{g})$ (equivalently, $\boldsymbol{\Theta}$) are already identified, and we have $T(\boldsymbol{\Theta})\boldsymbol{p} = T(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}} = T(\boldsymbol{\Theta})\bar{\boldsymbol{p}}$. Since $\mathbf{Q}$ contains a submatrix $\mathbf{I}_K$, the matrix $T(\boldsymbol{\Theta})$ has full column rank from a statement in Xu and Zhang (2016), and hence we obtain $\boldsymbol{p} = \bar{\boldsymbol{p}}$. This means all the parameters $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p})$ are identifiable as long as $\boldsymbol{p}$ satisfies (S.9). More precisely, we have that the DINA model parameters are identifiable if $(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{p}) \in \mathcal{T} \setminus \mathcal{N}_{R,1}$ where the set $\mathcal{N}_{R,1}$ is defined by (8) in the main text in Theorem 4. We rewrite the definition of $\mathcal{N}_{R,1}$, The above set $\mathcal{N}_{R,1}$ has measure zero with respect to the Lebesgue measure defined on the parameter space $\mathcal{T}$. This is because $\mathcal{N}_{R,1}$ is characterized by the zero set of a polynomial equation about entries of $\boldsymbol{p}$, and by basic algebraic geometry, $\mathcal{N}_{R,1}$ necessarily has measure zero in the parameter space of $\boldsymbol{p}$. This completes the proof of Theorem 1.

Proof of Theorem 4. We next examine the statistical interpretation of the null set $\mathcal{N}_{R,1}$ defined in (8) where identifiability breaks down. Recall the definition of the population proportion parameter $p_{\boldsymbol{\alpha}} = \mathbb{P}(\mathbf{A} = \boldsymbol{\alpha})$, where $\mathbf{A} = (A_1, \ldots, A_K)$ denotes a random attribute profile. For an arbitrary attribute pattern $\boldsymbol{\alpha} = (\alpha_1, \boldsymbol{\alpha}^*)$ where the subvector satisfies $\boldsymbol{\alpha}^* \in \{0,1\}^{K-1}$ and $\boldsymbol{\alpha}^* \succeq \boldsymbol{u}$, we have

$$
\begin{aligned}
&\mathbb{P}(A_1 = \alpha_1)\mathbb{P}(\mathbf{A}_{2:K} = \boldsymbol{\alpha}^*) \\
&= \Big(\sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(\alpha_1, \boldsymbol{\beta})}\Big)\big(p_{(\alpha_1, \boldsymbol{\alpha}^*)} + p_{(1-\alpha_1, \boldsymbol{\alpha}^*)}\big) \\
&= \sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(\alpha_1, \boldsymbol{\beta})} p_{(\alpha_1, \boldsymbol{\alpha}^*)} + \sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(\alpha_1, \boldsymbol{\beta})} p_{(1-\alpha_1, \boldsymbol{\alpha}^*)} \\
&= \sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(\alpha_1, \boldsymbol{\beta})} p_{(\alpha_1, \boldsymbol{\alpha}^*)} + \sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(1-\alpha_1, \boldsymbol{\beta})} p_{(\alpha_1, \boldsymbol{\alpha}^*)} \quad (\text{because } \boldsymbol{p} \in \mathcal{N}_{R,1}) \\
&= \Big(\sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(\alpha_1, \boldsymbol{\beta})} + \sum_{\boldsymbol{\beta}\in\{0,1\}^{K-1}} p_{(1-\alpha_1, \boldsymbol{\beta})}\Big) p_{(\alpha_1, \boldsymbol{\alpha}^*)} \\
&= p_{(\alpha_1, \boldsymbol{\alpha}^*)} = \mathbb{P}(\mathbf{A} = \boldsymbol{\alpha}).
\end{aligned}
$$

The third equality above follows from the fact that for $\boldsymbol{p} \in \mathcal{N}_{R,1}$, the $p_{(\alpha_1, \boldsymbol{\beta})} p_{(1-\alpha_1, \boldsymbol{\alpha}^*)} = p_{(1-\alpha_1, \boldsymbol{\beta})} p_{(\alpha_1, \boldsymbol{\alpha}^*)}$ holds for any $\alpha_1 \in \{0,1\}$ and $\boldsymbol{\alpha}^*, \boldsymbol{\beta} \in \{0,1\}^{K-1}$. Now we obtain that if

$\boldsymbol{p} \in \mathcal{N}_{R,1}$, then $\mathbb{P}(\mathbf{A} = (\alpha_1, \boldsymbol{\alpha}^*)) = \mathbb{P}(A_1 = \alpha_1)\mathbb{P}(\mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)$ for any $\alpha_1 \in \{0,1\}$ and $\boldsymbol{\alpha}^* \succeq \boldsymbol{u}$. This implies if $\boldsymbol{p} \in \mathcal{N}_{R,1}$, then latent attribute A_1 is conditionally independent of latent attributes $\mathbf{A}_{2:K}$ given $\mathbf{A}_{2:K} \succeq \boldsymbol{u}$.

On the other hand, if latent variables A_1 and $\mathbf{A}_{2:K}$ are conditionally independent given $\mathbf{A}_{2:K} \succeq \boldsymbol{u}$, then for any $\boldsymbol{\alpha}^* \succeq \boldsymbol{u}$ we have

$$\frac{p_{(1,\boldsymbol{\alpha}^*)}}{p_{(0,\boldsymbol{\alpha}^*)}} = \frac{\mathbb{P}(\mathbf{A} = (1, \boldsymbol{\alpha}^*))}{\mathbb{P}(\mathbf{A} = (0, \boldsymbol{\alpha}^*))} = \frac{\mathbb{P}(A_1 = 1)\mathbb{P}(\mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)}{\mathbb{P}(A_1 = 0)\mathbb{P}(\mathbf{A}_{2:K} = \boldsymbol{\alpha}^*)} = \frac{\mathbb{P}(A_1 = 1)}{\mathbb{P}(A_1 = 0)} =: \rho.$$

This means for any $\boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^*$ with $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$, the equality $p_{(1,\boldsymbol{\alpha}_1^*)}/p_{(0,\boldsymbol{\alpha}_1^*)} - p_{(1,\boldsymbol{\alpha}_2^*)}/p_{(0,\boldsymbol{\alpha}_2^*)} = \rho - \rho = 0$ must hold, which is equivalent to $p_{(1,\boldsymbol{\alpha}_1^*)}p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)}p_{(1,\boldsymbol{\alpha}_2^*)} = 0$ for any $\boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^*$ with $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}$. This means if $A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u}$ holds, then we must have $\boldsymbol{p} \in \mathcal{N}_{R,1}$ with $\mathcal{N}_{R,1}$ defined in (8) in Theorem 4.

Now we have proved the statement that

$$A_1 \perp\!\!\!\perp \mathbf{A}_{2:K} \mid \mathbf{A}_{2:K} \succeq \boldsymbol{u},$$

is exactly equivalent to the statement that

$$\boldsymbol{p} \in \mathcal{N}_{R,1} = \{p_{(1,\boldsymbol{\alpha}_1^*)}p_{(0,\boldsymbol{\alpha}_2^*)} - p_{(0,\boldsymbol{\alpha}_1^*)}p_{(1,\boldsymbol{\alpha}_2^*)} = 0 \text{ holds for any } \boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^* \text{ with } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}\}.$$

This completes the proof of Theorem 4. □

S.4 Proof of Theorem 2 and Theorem 5

Proof of Theorem 2. We rewrite the form of $\mathbf{Q}$ in (6) below,

$$\mathbf{Q} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_1 \\ \hline \mathbf{0} & \mathbf{Q}^{(1)} \end{array}\right), \quad \mathbf{Q}^{(1)} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_2 \\ \hline \mathbf{0} & \mathbf{Q}^{(2)} \end{array}\right), \quad \cdots, \quad \mathbf{Q}^{(m)} = \left(\begin{array}{cc} 1 & \mathbf{0} \\ 1 & \boldsymbol{u}_{m+1} \\ \hline \mathbf{0} & \mathbf{Q}^{(m+1)} \end{array}\right).$$

Under the assumption that the first $m+1$ latent attributes are each required by only two items, we know $\boldsymbol{u}_{1,\,1:m} = \mathbf{0}$, $\boldsymbol{u}_{2,\,1:(m-1)} = \mathbf{0}$, $\ldots$, $u_{m,\,1} = 0$. First consider the last $J-m-2$ items corresponding to the bottom $(J-m-2) \times K$ submatrix of $\mathbf{Q}$,

$$(\mathbf{0}, \quad \mathbf{Q}^{(m+1)}) =: \widetilde{\mathbf{Q}}^{(m+1)}$$

The $(J-m-2) \times (K-m-1)$ matrix $\mathbf{Q}^{(m+1)}$ satisfies the C-R-D conditions under the assumption stated in the corollary, and that the first $m+1$ columns of the $\widetilde{\mathbf{Q}}^{(m+1)}$ are all-zero columns. Next we use an argument similar to the proof of Theorem 1. Consider a true set of parameters $(\boldsymbol{\Theta}, \boldsymbol{p})$ and an alternative set $(\bar{\boldsymbol{\Theta}}, \bar{\boldsymbol{p}})$ with $T(\boldsymbol{\Theta})\boldsymbol{p} = T(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}}$. Then the following equations must hold for an arbitrary fixed response pattern $\boldsymbol{r} = (r_1, \ldots, r_{m+2}, \boldsymbol{r}^*)$,

$$\begin{aligned}
&\sum_{\boldsymbol{\alpha}^* \in \{0,1\}^{K-m-2}} \prod_{j > m+2:\, r_j = 1} \theta_{j,\,(\mathbf{0}, \boldsymbol{\alpha}^*)} \cdot \mathbb{P}(\mathbf{R}_{1:(m+2)} \geq \boldsymbol{r}_{1:(m+2)},\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*) \\
= &\sum_{\boldsymbol{\alpha}^* \in \{0,1\}^{K-m-2}} \prod_{j > m+2:\, r_j = 1} \bar{\theta}_{j,\,(\mathbf{0}, \boldsymbol{\alpha}^*)} \cdot \overline{\mathbb{P}}(\mathbf{R}_{1:(m+2)} \geq \boldsymbol{r}_{1:(m+2)},\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*).
\end{aligned}$$

Similar to the argument in the proof of Theorem 1, the fact that $\mathbf{Q}^{(m)}$ satisfies the C-R-D conditions imply $\boldsymbol{c}_{(J-m-1):J} = \bar{\boldsymbol{c}}_{(J-m-1):J}$ and $\boldsymbol{g}_{(J-m-1):J} = \bar{\boldsymbol{g}}_{(J-m-1):J}$, and also imply the following for all $\boldsymbol{\alpha}^* \in \{0,1\}^{K-m-2}$,

$$\mathbb{P}(\mathbf{R}_{1:(m+2)} \geq \boldsymbol{r}_{1:(m+2)},\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*) = \overline{\mathbb{P}}(\mathbf{R}_{1:(m+2)} \geq \boldsymbol{r}_{1:(m+2)},\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*). \quad \text{(S.10)}$$

Define surrogate (grouped) proportion parameters to be

$$p^{(m)}_{(z, \boldsymbol{\alpha}^*)} = \mathbb{P}(A_{m+1} = z,\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*), \quad z = 0, 1; \quad \text{(S.11)}$$

and define $\bar{p}^{(m)}_{(z, \boldsymbol{\alpha}^*)}$ similarly based on the alternative set of parameters $(\bar{\boldsymbol{\Theta}}, \bar{\boldsymbol{p}})$. Now fixing $(r_1, \ldots, r_m)^\top = \mathbf{0}$ and varying $(r_{m+1}, r_{m+2}) \in \{0,1\}^2$, the equality in (S.10) becomes

$$\mathbb{P}((R_{m+1}, R_{m+2}) \geq (r_{m+1}, r_{m+2}),\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*)$$

$$= \overline{\mathbb{P}}((R_{m+1}, R_{m+2}) \geq (r_{m+1}, r_{m+2}),\ \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*).$$

This implies the following equations for any fixed $\boldsymbol{\alpha}^* \succeq \boldsymbol{u}^{(m+1)}$ when (r_{m+1}, r_{m+2}) vary,

$$(r_{m+1}, r_{m+2}) = \begin{cases} (0,0) \implies p^{(m)}_{(0,\boldsymbol{\alpha}^*)} + p^{(m)}_{(1,\boldsymbol{\alpha}^*)} = \bar{p}^{(m)}_{(0,\boldsymbol{\alpha}^*)} + \bar{p}^{(m)}_{(1,\boldsymbol{\alpha}^*)}; \\ (1,0) \implies g_{m+1} \cdot p^{(m)}_{(0,\boldsymbol{\alpha}^*)} + c_{m+1} \cdot p^{(m)}_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_{m+1} \cdot \bar{p}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_{m+1} \cdot \bar{p}^{(m)}_{(1,\boldsymbol{\alpha}^*)}; \\ (0,1) \implies g_{m+2} \cdot p^{(m)}_{(0,\boldsymbol{\alpha}^*)} + c_{m+2} \cdot p^{(m)}_{(1,\boldsymbol{\alpha}^*)} = \bar{g}_{m+2} \cdot \bar{p}^{(m)}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_{m+2} \cdot \bar{p}^{(m)}_{(1,\boldsymbol{\alpha}^*)}; \\ (1,1) \implies g_{m+1}g_{m+2} \cdot p^{(m)}_{(0,\boldsymbol{\alpha}^*)} + c_{m+1}c_{m+2} \cdot p^{(m)}_{(1,\boldsymbol{\alpha}^*)} \\ \qquad\qquad\quad = \bar{g}_{m+1}\bar{g}_{m+2} \cdot \bar{p}^{(m)}_{(0,\boldsymbol{\alpha}^*)} + \bar{c}_{m+1}\bar{c}_{m+2} \cdot \bar{p}^{(m)}_{(1,\boldsymbol{\alpha}^*)}. \end{cases} \tag{S.12}$$

The above system of four equations are similar in form to Eq. (S.7) in the proof of Theorem 1. So following a similar argument as before, we obtain that (c_{m+1}, c_{m+2}) and (g_{m+1}, g_{m+2}) and all the $p^{(m)}_{(z,\boldsymbol{\alpha}^*)}$'s are identifiable as long as $\boldsymbol{p} \in \mathcal{T} \setminus \mathcal{N}_m$ where

$$\mathcal{N}_m = \{p^{(m)}_{(1,\boldsymbol{\alpha}_1^*)} p^{(m)}_{(0,\boldsymbol{\alpha}_2^*)} - p^{(m)}_{(0,\boldsymbol{\alpha}_1^*)} p^{(m)}_{(1,\boldsymbol{\alpha}_2^*)} = 0 \text{ for any } \boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^* \text{ with } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}^{(m+1)}\}. \tag{S.13}$$

Note the definition (S.11) implies that each surrogate proportion $p^{(m)}_{(z,\boldsymbol{\alpha}^*)}$ is a sum of certain individual proportion parameters in that

$$p^{(m)}_{(z,\boldsymbol{\alpha}^*)} = \sum_{\boldsymbol{\beta} \in \{0,1\}^m} p_{(\boldsymbol{\beta},z,\boldsymbol{\alpha}^*)}.$$

Note that the $p^{(m)}_{(z,\boldsymbol{\alpha}^*)}$ defined above exactly characterizes the joint distribution of latent attributes A_m and $\mathbf{A}_{(m+1):K}$. Now we have that the set $\mathcal{N}_m$ defined in (S.13) corresponds to the zero set of certain polynomials about the proportion parameters $\boldsymbol{p}$, so $\mathcal{N}_m$ has Lebesgue measure zero in the parameter space $\mathcal{T}$. Therefore we have shown (c_{m+1}, c_{m+2}), (g_{m+1}, g_{m+2}), and $\boldsymbol{p}^{(m)} := (p^{(m)}_{(z,\boldsymbol{\alpha}^*)};\ (z, \boldsymbol{\alpha}^*) \in \{0,1\}^{K-m})$ are generically identifiable.

Moreover, we go back to the equality in (S.10) and define surrogate proportions alternatively as

$$p^{(m),\boldsymbol{r}}_{(z,\boldsymbol{\alpha}^*)} = \mathbb{P}(\mathbf{R}_{1:m} \succeq \boldsymbol{r}_{1:m},\, A_{m+1} = z,\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*), \quad x = 0, 1;$$

and define $\bar{p}^{(m),\boldsymbol{r}}_{(z,\boldsymbol{\alpha}^*)}$ similarly. Fixing $\boldsymbol{r}_{1:m}$ and varying $(r_{m+1}, r_{m+2}) \in \{0,1\}^2$, Eq. (S.10) can be written in a similar form as the four equations in (S.12), with $p^{(m)}_{(z,\boldsymbol{\alpha}^*)}$ there replaced by $p^{(m),\boldsymbol{r}}_{(z,\boldsymbol{\alpha}^*)}$ now. Since when $\boldsymbol{p} \in \mathcal{T} \setminus \mathcal{N}_m$, we already have the item parameters (c_{m+1}, c_{m+2}) and (g_{m+1}, g_{m+2}) are identifiable, based on the equations about (c_{m+1}, c_{m+2}), (g_{m+1}, g_{m+2}), and $\boldsymbol{p}^{(m),\boldsymbol{r}}$, the parameters $\boldsymbol{p}^{(m),\boldsymbol{r}}$ are also identifiable. Now we write out the equality $\boldsymbol{p}^{(m),\boldsymbol{r}} = \bar{\boldsymbol{p}}^{(m),\boldsymbol{r}}$ by their definitions as

$$\mathbb{P}(\mathbf{R}_{1:m} \geq \boldsymbol{r}_{1:m},\, A_{m+1} = z,\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*) = \overline{\mathbb{P}}(\mathbf{R}_{1:m} \geq \boldsymbol{r}_{1:m},\, A_{m+1} = z,\, \mathbf{A}_{(m+2):K} = \boldsymbol{\alpha}^*),$$

where $(z, \boldsymbol{\alpha}^*) \in \{0,1\}^{K-m}$. Therefore the above equation can be equivalently written as follows, with the new $\boldsymbol{\alpha}^*$ defined to be $(K-m)$-dimensional,

$$\mathbb{P}(\mathbf{R}_{1:m} \geq \boldsymbol{r}_{1:m},\, \mathbf{A}_{(m+1):K} = \boldsymbol{\alpha}^*) = \overline{\mathbb{P}}(\mathbf{R}_{1:m} \geq \boldsymbol{r}_{1:m},\, \mathbf{A}_{(m+1):K} = \boldsymbol{\alpha}^*). \tag{S.14}$$

Comparing the above (S.14) to the previous (S.10) give an immediate similarity, with the difference being only the changes of subscripts of $\mathbf{R}$ and $\mathbf{A}$. Therefore, we can proceed in the same way as before, and show the identifiability of (c_{m-1}, c_m) and (g_{m-1}, g_m) and all the $p^{(m-1)}_{(z,\boldsymbol{\alpha}^*)}$ when $\boldsymbol{p}$ satisfies $\boldsymbol{p} \in \mathcal{T} \setminus (\mathcal{N}_m \cup \mathcal{N}_{m-1})$, where

$$\mathcal{N}_{m-1} = \{p^{(m-1)}_{(1,\boldsymbol{\alpha}_1^*)} p^{(m-1)}_{(0,\boldsymbol{\alpha}_2^*)} - p^{(m-1)}_{(0,\boldsymbol{\alpha}_1^*)} p^{(m-1)}_{(1,\boldsymbol{\alpha}_2^*)} = 0 \text{ for any } \boldsymbol{\alpha}_1^* \neq \boldsymbol{\alpha}_2^* \text{ with } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}^{(m)} \vee (0, \boldsymbol{u}^{(m+1)})\}.$$

In the definition of $\mathcal{N}_{m-1}$, we have $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}^{(m)} \vee (0, \boldsymbol{u}^{(m+1)}) = \widetilde{\boldsymbol{u}}^{(m)} \vee \widetilde{\boldsymbol{u}}^{(m+1)}$ because the $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*$ first need to satisfy the previous requirement before (S.12) and hence $\boldsymbol{\alpha}_{1,-1}^*, \boldsymbol{\alpha}_{2,-1}^* \succeq \boldsymbol{u}^{(m+1)}$ (equivalently, $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq (0, \boldsymbol{u}^{(m+1)})$); and additionally they also need to satisfy the new requirement $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \boldsymbol{u}^{(m)}$.

Recall the definition that $\widetilde{\boldsymbol{u}}^{(\ell)} = (\mathbf{0}, \boldsymbol{u}^{(\ell)})$ is a $(K-1)$-dimensional vector for $\ell = 2, \ldots, m+1$, and $\widetilde{\boldsymbol{u}}^{(1)} = \boldsymbol{u}^{(1)}$ is also a $(K-1)$-dimensional vector. Proceeding in an iterative manner as done in the previous paragraphs, we obtain that as long as $\boldsymbol{p}$ satisfies the following condition, then all the item parameters $\boldsymbol{c}$, $\boldsymbol{g}$ and all the proportion parameters $\boldsymbol{p}$ are identifiable.

$$\boldsymbol{p} \in \mathcal{T} \setminus \left\{ \bigcup_{\ell=0}^{m} \mathcal{N}_\ell \right\},$$

$$\mathcal{N}_\ell = \left\{ p^{(\ell)}_{(1,\boldsymbol{\alpha}_1^*)} p^{(\ell)}_{(0,\boldsymbol{\alpha}_2^*)} - p^{(\ell)}_{(0,\boldsymbol{\alpha}_1^*)} p^{(\ell)}_{(1,\boldsymbol{\alpha}_2^*)} = 0 \text{ for any } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \succeq \bigvee_{t=\ell+1}^{m+1} \widetilde{\boldsymbol{u}}^{(t)} \right\};$$

$$\text{with the definition } \; p^{(\ell)}_{(z,\boldsymbol{\alpha}^*)} = \mathbb{P}(A_{\ell+1} = z, \, \mathbf{A}_{(\ell+2):K} = \boldsymbol{\alpha}^*),$$

Because of the assumption

$$\bigvee_{t=1}^{m+1} \widetilde{\boldsymbol{u}}^{(t)} \neq \mathbf{1}_{K-1}^\top \tag{S.15}$$

stated in the theorem, we claim that the set $\mathcal{T} \setminus \{\bigcup_{\ell=0}^m \mathcal{N}_\ell\}$ is nonempty. To see this, note that $\bigvee_{t=\ell+1}^{m+1} \widetilde{\boldsymbol{u}}^{(t)} \neq \mathbf{1}_{K-1}^\top$ for each $\ell = 0, \ldots, m$ follows from (S.15). This means there must exist two distinct patterns $\boldsymbol{\alpha}_{1,\ell}^* \neq \boldsymbol{\alpha}_{2,\ell}^*$ with $\boldsymbol{\alpha}_{1,\ell}^*$, $\boldsymbol{\alpha}_{2,\ell}^* \succeq \bigvee_{t=\ell+1}^{m+1} \widetilde{\boldsymbol{u}}^{(t)}$. Therefore as long as $\boldsymbol{p}$ satisfies $p^{(\ell)}_{(1,\boldsymbol{\alpha}_{1,\ell}^*)} p^{(\ell)}_{(0,\boldsymbol{\alpha}_{2,\ell}^*)} - p^{(\ell)}_{(0,\boldsymbol{\alpha}_{1,\ell}^*)} p^{(\ell)}_{(1,\boldsymbol{\alpha}_{2,\ell}^*)} \neq 0$ for each $\ell = 0, \ldots, m$, such $\boldsymbol{p}$ does not belong to $\bigcup_{\ell=0}^m \mathcal{N}_\ell$ and hence $\boldsymbol{p} \in \mathcal{T} \setminus \{\bigcup_{\ell=0}^m \mathcal{N}_\ell\}$. This proves the earlier claim that the subset of the identifiable parameters $\mathcal{T} \setminus \{\bigcup_{\ell=0}^m \mathcal{N}_\ell\}$ is nonempty.

Now note that the subset of the parameter space where identifiability may break down $\bigcup_{\ell=0}^m \mathcal{N}_\ell$ is a finite union of several zero sets of polynomial equations about entries of $\boldsymbol{p}$, so it necessarily has Lebesgue measure zero in $\mathcal{T}$. This proves the generic identifiability of parameters $(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{p})$ and completes the proof of Theorem 2. Furthermore, note that the $\mathcal{N}_\ell$ in the last paragraph gives the form of the non-identifiable null sets in Theorem 5. Recall that the notation $p^{(\ell)}_{(z,\boldsymbol{\alpha}^*)}$ exactly corresponds to the marginal distribution of the $K - \ell$ latent attributes $A_{\ell+1}, \ldots, A_K$. So each set $\mathcal{N}_\ell$ can be equivalently written as

$$\mathcal{N}_\ell = \left\{ A_\ell \perp\!\!\!\perp \mathbf{A}_{(\ell+1):K} \;\middle|\; \left\{ \mathbf{A}_{(\ell+1):K} \succeq \bigvee_{t=\ell+1}^{m+1} \widetilde{\boldsymbol{u}}^{(t)} \right\} \right\}.$$

The above set $\mathcal{N}_\ell$ carries the statistical interpretation of latent conditional independence. This completes the proof Theorem 5. □

S.5 Proof of Theorem 3 and Theorem 6

We rewrite the form of the $\mathbf{Q}$-matrix in the theorem below,

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \boldsymbol{v} & \boldsymbol{v} & \mathbf{Q}^\star \end{pmatrix} = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{Q}' \\ \mathbf{1} & \mathbf{1} & \mathbf{Q}'' \end{pmatrix}.$$

Denote the size of the above submatrix $\mathbf{Q}'$ by $J_1 \times (K-2)$, then $\mathbf{Q}''$ has size $(J-2-J_1) \times (K-2)$. By Remark 4, we have $J-2-J_1 \geq 2$. Consider two sets of DINA model parameters $(\boldsymbol{c}, \boldsymbol{g}, \boldsymbol{p})$ and $(\bar{\boldsymbol{c}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{p}})$ that lead to the same distribution of $\mathbf{R}$ so we have $T(\boldsymbol{\Theta})\boldsymbol{p} = T(\bar{\boldsymbol{\Theta}})\bar{\boldsymbol{p}}$. Theorem 4 in Xu and Zhang (2016) established that if $\mathbf{Q}$ satisfies Conditions (C) and (R), then the guessing parameters associated with those items requiring more than one attribute (i.e., $\{g_j : \sum_{k=1}^K q_{j,k} > 1\}$) and all the slipping parameters (i.e., $\{c_1, \ldots, c_J\}$) are identifiable. Since the considered $\mathbf{Q}$-matrix satisfies Conditions (C) and (R) by the assumption in the theorem, we have $\boldsymbol{c} = \bar{\boldsymbol{c}}$ and $\boldsymbol{g}_{(3+J_1):J} = \bar{\boldsymbol{g}}_{(3+J_1):J}$.

Next consider an arbitrary $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$. The form of the $\mathbf{Q}$-matrix implies

$$\theta_{j,\,(0,0,\boldsymbol{\alpha}^*)} = \theta_{j,\,(0,1,\boldsymbol{\alpha}^*)} = \theta_{j,\,(1,0,\boldsymbol{\alpha}^*)} = \theta_{j,\,(1,1,\boldsymbol{\alpha}^*)}, \quad \forall j \in \{2, \ldots, 2+J_1\}.$$

So for a response pattern $\boldsymbol{r}$ with $\boldsymbol{r}_{(3+J_1):J} = \mathbf{0}$, we can write $T_{\boldsymbol{r},:}(\boldsymbol{\Theta})\boldsymbol{p}$ as follows,

$$
\begin{aligned}
& T_{\boldsymbol{r},:}(\boldsymbol{\Theta})\boldsymbol{p} \\
= & \sum_{\substack{\boldsymbol{\alpha}\in\{0,1\}^K \\ \boldsymbol{\alpha}=(\alpha_1,\alpha_2,\boldsymbol{\alpha}^*)}} p_{\boldsymbol{\alpha}} \cdot \mathbb{P}(R_1 \geq r_1, R_2 \geq r_2 \mid \mathbf{A} = \boldsymbol{\alpha}) \prod_{j=3}^{2+J_1} \theta_{j,(0,0,\boldsymbol{\alpha}^*)} \\
= & \sum_{\boldsymbol{\alpha}^*\in\{0,1\}^{K-2}} \underbrace{\left[\sum_{(\alpha_1,\alpha_2)\in\{0,1\}^2} p_{(\alpha_1,\alpha_2,\boldsymbol{\alpha}^*)} \cdot \mathbb{P}(R_1 \geq r_1, R_2 \geq r_2 \mid \mathbf{A}_{1:2} = (\alpha_1,\alpha_2))\right]}_{\text{define this to be } p_{\boldsymbol{\alpha}^*}^{(r_1,r_2)}} \prod_{j=3}^{2+J_1} \theta_{j,(0,0,\boldsymbol{\alpha}^*)}.
\end{aligned}
$$

Now define surrogate DINA model parameters: surrogate proportions $\boldsymbol{p}^{(r_1,r_2)} = (p_{\boldsymbol{\alpha}^*}^{(r_1,r_2)} : \boldsymbol{\alpha}^* \in \{0,1\}^{K-2})$ and surrogate item parameters $\boldsymbol{\Theta}^* = \{\theta_{j,(0,0,\boldsymbol{\alpha}^*)} : \ j = 3,\ldots,2+J_1;\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\}$. These surrogate parameters $\boldsymbol{p}^{(r_1,r_2)}$ and $\boldsymbol{\Theta}^*$ can be viewed as associated with the $J_1 \times (K-2)$ matrix $\mathbf{Q}'$. Since $\mathbf{Q}'$ satisfies the C-R-D conditions, we obtain the identifiability of $\boldsymbol{p}^{(r_1,r_2)}$ and $\boldsymbol{\Theta}^*$. Note that $\boldsymbol{\Theta}^*$ includes all the item parameters associated with items with indices $3,\ldots,J$; i.e., we have established the identifiability of $\{c_3,\ldots,c_{2+J_1},g_3,\ldots,g_{2+J_1}\}$. So far we have obtained $\boldsymbol{c} = \bar{\boldsymbol{c}}$ and $\boldsymbol{g}_{3:J} = \bar{\boldsymbol{g}}_{3:J}$. It only remains to identify $\boldsymbol{p}$ and (g_1,g_2).

The identifiability of $\boldsymbol{p}^{(r_1,r_2)}$ means $\boldsymbol{p}^{(r_1,r_2)} = \bar{\boldsymbol{p}}^{(r_1,r_2)}$ for $(r_1,r_2) \in \{0,1\}^2$, which gives

$$
(r_1,r_2) = \begin{cases}
(0,0): \ p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)} + p_{(0,1,\boldsymbol{\alpha}^*)} + p_{(1,1,\boldsymbol{\alpha}^*)} \\
\qquad\qquad = \bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(0,1,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,1,\boldsymbol{\alpha}^*)}; \\
(1,0): \ g_1[p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(0,1,\boldsymbol{\alpha}^*)}] + c_1[p_{(1,0,\boldsymbol{\alpha}^*)} + p_{(1,1,\boldsymbol{\alpha}^*)}] \\
\qquad\qquad = \bar{g}_1[\bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(0,1,\boldsymbol{\alpha}^*)}] + c_1[\bar{p}_{(1,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,1,\boldsymbol{\alpha}^*)}]; \\
(0,1): \ g_2[p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)}] + c_2[p_{(0,1,\boldsymbol{\alpha}^*)} + p_{(1,1,\boldsymbol{\alpha}^*)}] \\
\qquad\qquad = \bar{g}_2[\bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,0,\boldsymbol{\alpha}^*)}] + c_2[\bar{p}_{(0,1,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,1,\boldsymbol{\alpha}^*)}]; \\
(1,1): \ g_1 g_2 p_{(0,0,\boldsymbol{\alpha}^*)} + c_1 g_2 p_{(1,0,\boldsymbol{\alpha}^*)} + g_1 c_2 p_{(0,1,\boldsymbol{\alpha}^*)} + c_1 c_2 p_{(1,1,\boldsymbol{\alpha}^*)} \\
\qquad\qquad = \bar{g}_1\bar{g}_2\bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + c_1\bar{g}_2\bar{p}_{(1,0,\boldsymbol{\alpha}^*)} + \bar{g}_1 c_2\bar{p}_{(0,1,\boldsymbol{\alpha}^*)} + c_1 c_2\bar{p}_{(1,1,\boldsymbol{\alpha}^*)}.
\end{cases}
\tag{S.16}
$$

Since $\mathbf{Q}'$ satisfies Condition (C) and contains a submatrix $\mathbf{I}_{K-2}$, we can assume without loss of generality that the first $K-2$ rows of $\mathbf{Q}'$ form $\mathbf{I}_{K-2}$; namely, the first K rows of $\mathbf{Q}$ forms an identity matrix $\mathbf{I}_K$. According to the form of $\mathbf{Q}$, let $\boldsymbol{q}_m = (1, 1, 0, \dots, 0)$ for some $m \in \{3 + J_1, \dots, J\}$. Given an arbitrary pattern $\boldsymbol{\alpha}^* = (\alpha_3, \dots, \alpha_K) \in \{0, 1\}^{K-2}$, define

$$\boldsymbol{\theta}^* = \sum_{\substack{3\leq k\leq K:\\ \alpha_k=1}} g_k \boldsymbol{e}_k + \sum_{\substack{3\leq k\leq K:\\ \alpha_k=0}} c_k \boldsymbol{e}_k + g_m \boldsymbol{e}_m.$$

Then $T_{\boldsymbol{r},:}(\boldsymbol{\Theta} - \boldsymbol{\theta}^* \cdot \mathbf{1}_{2^K})\boldsymbol{p} = T_{\boldsymbol{r},:}(\bar{\boldsymbol{\Theta}} - \boldsymbol{\theta}^* \cdot \mathbf{1}_{2^K})\bar{\boldsymbol{p}}$ gives

$$\begin{aligned} & p_{(1,1,\boldsymbol{\alpha}^*)} \prod_{\substack{3\leq k\leq K:\\ \alpha_k=1}} (c_k - g_k) \prod_{\substack{3\leq k\leq K:\\ \alpha_k=0}} (g_k - c_k)(c_m - g_m) \\ = & \bar{p}_{(1,1,\boldsymbol{\alpha}^*)} \prod_{\substack{3\leq k\leq K:\\ \alpha_k=1}} (c_k - g_k) \prod_{\substack{3\leq k\leq K:\\ \alpha_k=0}} (g_k - c_k)(c_m - g_m), \end{aligned}$$

which implies $p_{(1,1,\boldsymbol{\alpha}^*)} = \bar{p}_{(1,1,\boldsymbol{\alpha}^*)}$. Note that this identifiability conclusion holds for any $\boldsymbol{\alpha}^* \in \{0, 1\}^K$. Plugging the $p_{(1,1,\boldsymbol{\alpha}^*)} = \bar{p}_{(1,1,\boldsymbol{\alpha}^*)}$ into (S.16) gives the following equations about undetermined parameters $\bar{g}_1$, $\bar{g}_2$, and $\{p_{(0,0,\boldsymbol{\alpha}^*)}, p_{(0,1,\boldsymbol{\alpha}^*)}, p_{(1,0,\boldsymbol{\alpha}^*)} : \boldsymbol{\alpha}^* \in \{0, 1\}^{K-2}\}$,

$$(r_1, r_2) = \begin{cases} (0,0) \implies p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)} + p_{(0,1,\boldsymbol{\alpha}^*)} = \bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(0,1,\boldsymbol{\alpha}^*)}; \\ (1,0) \implies g_1[p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(0,1,\boldsymbol{\alpha}^*)}] + c_1 p_{(1,0,\boldsymbol{\alpha}^*)} = \bar{g}_1[\bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(0,1,\boldsymbol{\alpha}^*)}] + c_1 \bar{p}_{(1,0,\boldsymbol{\alpha}^*)}; \\ (0,1) \implies g_2[p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)}] + c_2 p_{(0,1,\boldsymbol{\alpha}^*)} = \bar{g}_2[\bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + \bar{p}_{(1,0,\boldsymbol{\alpha}^*)}] + c_2 \bar{p}_{(0,1,\boldsymbol{\alpha}^*)}; \\ (1,1) \implies g_1 g_2 p_{(0,0,\boldsymbol{\alpha}^*)} + c_1 g_2 p_{(1,0,\boldsymbol{\alpha}^*)} + g_1 c_2 p_{(0,1,\boldsymbol{\alpha}^*)} \\ \qquad\qquad = \bar{g}_1 \bar{g}_2 \bar{p}_{(0,0,\boldsymbol{\alpha}^*)} + c_1 \bar{g}_2 \bar{p}_{(1,0,\boldsymbol{\alpha}^*)} + \bar{g}_1 c_2 \bar{p}_{(0,1,\boldsymbol{\alpha}^*)}. \end{cases} \tag{S.17}$$

After some transformation, (S.17) yields

$$\begin{cases}(g_1-\bar{g}_1)(p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)})+(c_1-\bar{g}_1)p_{(1,0,\boldsymbol{\alpha}^*)}=(c_1-\bar{g}_1)\bar{p}_{(1,0,\boldsymbol{\alpha}^*)},\\ (g_1-\bar{g}_1)(g_2-c_2)p_{(0,0,\boldsymbol{\alpha}^*)}+(c_1-\bar{g}_1)(g_2-c_2)p_{(1,0,\boldsymbol{\alpha}^*)}=(c_1-\bar{g}_1)(\bar{g}_2-c_2)\bar{p}_{(1,0,\boldsymbol{\alpha}^*)}.\end{cases} \tag{S.18}$$

The right hand sides of both of the above equations are nonzero. So we can take the ratio of these two equations to obtain

$$\frac{(g_1-\bar{g}_1)p_{(0,0,\boldsymbol{\alpha}^*)}/p_{(1,0,\boldsymbol{\alpha}^*)}+(c_1-\bar{g}_1)}{(g_1-\bar{g}_1)[p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)}]/p_{(1,0,\boldsymbol{\alpha}^*)}+(c_1-\bar{g}_1)}(g_2-c_2)=\bar{g}_2-c_2.$$

Define $f(\boldsymbol{\alpha}^*) = p_{(0,0,\boldsymbol{\alpha}^*)}/p_{(1,0,\boldsymbol{\alpha}^*)}$, $g(\boldsymbol{\alpha}^*) = [p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)}]/p_{(1,0,\boldsymbol{\alpha}^*)}$ as functions of $\boldsymbol{\alpha}^*$, then the above equation can be written as

$$\frac{A\cdot f(\boldsymbol{\alpha}^*)+B}{A\cdot g(\boldsymbol{\alpha}^*)+B}=C,$$

where $A=g_1-\bar{g}_1$, $B=c_1-\bar{g}_1$, and $C=\bar{g}_2-c_2$. So we have

$$A\cdot(f(\boldsymbol{\alpha}^*)-C\cdot g(\boldsymbol{\alpha}^*))=BC-B,$$

which is equivalent to

$$(g_1-\bar{g}_1)\cdot\left[\frac{p_{(0,0,\boldsymbol{\alpha}^*)}}{p_{(1,0,\boldsymbol{\alpha}^*)}}-(\bar{g}_2-c_2)\frac{p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)}}{p_{(1,0,\boldsymbol{\alpha}^*)}}\right]=(c_1-\bar{g}_1)(\bar{g}_2-c_2)-(c_1-\bar{g}_1).$$

Consider $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*$, we further obtain the following function $h(\boldsymbol{\alpha}^*)$ does not depend on $\boldsymbol{\alpha}^*$,

$$\begin{aligned}h(\boldsymbol{\alpha}^*) &:= (g_1-\bar{g}_1)\cdot\left[\frac{p_{(0,0,\boldsymbol{\alpha}^*)}}{p_{(1,0,\boldsymbol{\alpha}^*)}}-(\bar{g}_2-c_2)\frac{p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)}}{p_{(1,0,\boldsymbol{\alpha}^*)}}\right]\\ &= (g_1-\bar{g}_1)\cdot\frac{p_{(0,0,\boldsymbol{\alpha}^*)}+(c_2-\bar{g}_2)(p_{(0,0,\boldsymbol{\alpha}^*)}+p_{(0,1,\boldsymbol{\alpha}^*)})}{p_{(1,0,\boldsymbol{\alpha}^*)}};\end{aligned}$$

therefore we have

$$\begin{aligned}
0 &= h(\boldsymbol{\alpha}_1^*) - h(\boldsymbol{\alpha}_2^*)\\
&= (g_1 - \bar{g}_1)\cdot\Big[\frac{p_{(0,0,\boldsymbol{\alpha}_1^*)} + (c_2-\bar{g}_2)(p_{(0,0,\boldsymbol{\alpha}_1^*)} + p_{(0,1,\boldsymbol{\alpha}_1^*)})}{p_{(1,0,\boldsymbol{\alpha}_1^*)}}\\
&\qquad\qquad - \frac{p_{(0,0,\boldsymbol{\alpha}_2^*)} + (c_2-\bar{g}_2)(p_{(0,0,\boldsymbol{\alpha}_2^*)} + p_{(0,1,\boldsymbol{\alpha}_2^*)})}{p_{(1,0,\boldsymbol{\alpha}_2^*)}}\Big]\\
&= (g_1 - \bar{g}_1)\frac{1}{p_{(1,0,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)}}\Big\{[p_{(0,0,\boldsymbol{\alpha}_1^*)} + (c_2-\bar{g}_2)(p_{(0,0,\boldsymbol{\alpha}_1^*)} + p_{(0,1,\boldsymbol{\alpha}_1^*)})]p_{(1,0,\boldsymbol{\alpha}_2^*)}\\
&\qquad\qquad - [p_{(0,0,\boldsymbol{\alpha}_2^*)} + (c_2-\bar{g}_2)(p_{(0,0,\boldsymbol{\alpha}_2^*)} + p_{(0,1,\boldsymbol{\alpha}_2^*)})]p_{(1,0,\boldsymbol{\alpha}_1^*)}\Big\}.
\end{aligned}$$

According to the above equality, if $g_1 - \bar{g}_1 \neq 0$, then $h(\boldsymbol{\alpha}_1^*) - h(\boldsymbol{\alpha}_2^*) = 0$ gives

$$\begin{aligned}
&p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)}p_{(1,0,\boldsymbol{\alpha}_1^*)}\\
&+(c_2-\bar{g}_2)[(p_{(0,0,\boldsymbol{\alpha}_1^*)} + p_{(0,1,\boldsymbol{\alpha}_1^*)})p_{(1,0,\boldsymbol{\alpha}_2^*)} - (p_{(0,0,\boldsymbol{\alpha}_2^*)} + p_{(0,1,\boldsymbol{\alpha}_2^*)})p_{(1,0,\boldsymbol{\alpha}_1^*)}] = 0.
\end{aligned} \tag{S.19}$$

We rewrite below the definitions of the functions m_1, m_2, m_3 stated in (11) in the theorem,

$$\begin{cases}
m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,1,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,1,\boldsymbol{\alpha}_2^*)}p_{(1,0,\boldsymbol{\alpha}_1^*)},\\
m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)}p_{(1,0,\boldsymbol{\alpha}_1^*)},\\
m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(0,1,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)}p_{(0,1,\boldsymbol{\alpha}_1^*)}.
\end{cases}$$

Then (S.19) can be written as

$$m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) + (c_2-\bar{g}_2)[m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) + m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)] = 0. \tag{S.20}$$

Note that $c_2 - \bar{g}_2 \neq 0$. If $m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \neq 0$ holds for some $\boldsymbol{\alpha}_1^*$ and $\boldsymbol{\alpha}_2^*$, then we can obtain the following from (S.20),

$$\frac{m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)}{m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)} := \frac{p_{(0,1,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,1,\boldsymbol{\alpha}_2^*)}p_{(1,0,\boldsymbol{\alpha}_1^*)}}{p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)}p_{(1,0,\boldsymbol{\alpha}_1^*)}} = \frac{1}{\bar{g}_2 - c_2} - 1. \tag{S.21}$$

Therefore, as long as there exist $\boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}$ such that $\boldsymbol{p}$ satisfies

$$\frac{m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)}{m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)} \neq \frac{m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)}{m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)}, \quad m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \neq 0, \quad m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \neq 0,$$

then (S.21) cannot hold true; such a contradiction implies the earlier assumption $g_1 - \bar{g}_1 \neq 0$ is incorrect, and we should have $g_1 = \bar{g}_1$. Equivalently, we have shown that if there exist $\boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}$ such that

$$m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \neq 0, \quad m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \neq 0, \quad m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \neq 0,$$

then $g_1 = \bar{g}_1$ and hence parameter g_1 is identifiable.

Define a subset $\mathcal{N}_{D,1}$ of the parameter space $\mathcal{T}$ to be

$$\begin{aligned}
\mathcal{N}_{D,1} = \{&\text{For all } \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}, \\
&\quad \text{Either } \ m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) = 0, \\
&\quad \text{Or } \ m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = 0, \ \text{Or } \ m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) = 0.\} \\
= \{&\text{For all } \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}, \\
&\quad m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \cdot m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \cdot [m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_2(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)] = 0.\}.
\end{aligned}$$

Then we have established that as long as $\boldsymbol{p} \in \mathcal{T} \setminus \mathcal{N}_{D,1}$, then $g_1 = \bar{g}_1$ and parameter g_1 is identifiable. By the symmetry between g_1 and g_2, we similarly obtain that if $\boldsymbol{p} \in \mathcal{T} \setminus \mathcal{N}_{D,2}$, then $g_2 = \bar{g}_2$ and parameter g_2 is identifiable, where $\mathcal{N}_{D,2}$ takes the following form,

$$\begin{aligned}
\mathcal{N}_{D,2} = \{&\text{For all } \boldsymbol{\alpha}_1^*,\ \boldsymbol{\alpha}_2^*,\ \boldsymbol{\beta}_1^*,\ \boldsymbol{\beta}_2^* \in \{0,1\}^{K-2}, \\
&\quad m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) \cdot m_3(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) \cdot [m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_3(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*) - m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) m_1(\boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)] = 0.\}.
\end{aligned}$$

The function $m_3(\cdot,\cdot)$ has been defined earlier together with $m_1(\cdot,\cdot)$ and $m_2(\cdot,\cdot)$. In summary, if $\boldsymbol{p} \in \mathcal{T} \setminus (\mathcal{N}_{D,1} \cup \mathcal{N}_{D,2})$, then g_1 and g_2 are identifiable.

Recall that we previously have already proved the identifiability of all the other item parameters and also identifiability of $\{p_{(1,1,\boldsymbol{\alpha}^*)}: \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\}$. Now we can replace $\bar{g}_1$ by g_1 in the first equation in (S.18) and obtain $\bar{p}_{(1,0,\boldsymbol{\alpha}^*)} = p_{(1,0,\boldsymbol{\alpha}^*)}$; similarly, replacing $\bar{g}_2$ by g_2 in (S.17) gives $\bar{p}_{(0,1,\boldsymbol{\alpha}^*)} = p_{(0,1,\boldsymbol{\alpha}^*)}$. With $\bar{p}_{(1,0,\boldsymbol{\alpha}^*)}$ and $\bar{p}_{(0,1,\boldsymbol{\alpha}^*)}$ both determined, (S.17) finally gives $\bar{p}_{(1,1,\boldsymbol{\alpha}^*)} = p_{(1,1,\boldsymbol{\alpha}^*)}$. Noting that the above argument holds for an arbitrary $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$, we have established the identifiability of all the parameters under the DINA model under the condition that the true proportion parameters $\boldsymbol{p}$ satisfies $\boldsymbol{p} \in \mathcal{T} \setminus (\mathcal{N}_{D,1} \cup \mathcal{N}_{D,2})$. Note that the set $\mathcal{N}_{D,1} \cup \mathcal{N}_{D,2}$ where identifiability potentially breaks down is characterized by the zero sets of certain nontrivial polynomial equations about the entries of $\boldsymbol{p}$, and hence necessarily has Lebesgue measure zero in the parameter space $\mathcal{T}$. This proves the conclusion of generic identifiability and concludes the proof of Theorem 3. Further note that the forms of $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$ defined in the last paragraph are exactly the same as those stated in Theorem 6, so we have also proved Theorem 6. □

S.6 Proof of Proposition 3

We introduce some new notation to facilitate understanding the null sets $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$. Consider the joint distribution of two discrete random variables $Z_1 := (A_1, A_2)$ and $Z_2 := (A_3, \dots, A_K)$, each concatenated from the latent attributes. That is, Z_1 concatenates two variables A_1 and A_2 and takes $|\{0,1\}^2| = 4$ possible values, and Z_2 concatenates $K-2$ binary variables and takes $|\{0,1\}^{K-2}| = 2^{K-2}$ possible values. The joint distribution of Z_1 and Z_2 can be written in the form of a $4 \times 2^{K-2}$ contingency table, whose rows are indexed by the possible values Z_1 can take and columns by the possible values Z_2 can take. Each entry in this table represents the probability of a specific configuration of (Z_1, Z_2). We write out this $4 \times 2^{K-2}$ table below and denote it by $\mathcal{B}$,

$$\begin{array}{c} \\ (00) \\ (10) \\ (01) \\ (11) \end{array} \begin{array}{c} \begin{array}{cccc} (10\cdots0) & (01\cdots0) & \cdots & (11\cdots1) \end{array} \\ \begin{pmatrix} p_{(00,10\cdots0)} & p_{(00,01\cdots0)} & \cdots & p_{(00,11\cdots1)} \\ p_{(10,10\cdots0)} & p_{(10,01\cdots0)} & \cdots & p_{(10,11\cdots1)} \\ p_{(01,10\cdots0)} & p_{(01,01\cdots0)} & \cdots & p_{(01,11\cdots1)} \\ p_{(11,10\cdots0)} & p_{(11,01\cdots0)} & \cdots & p_{(11,11\cdots1)} \end{pmatrix} \end{array} \tag{S.22}$$

Note that when the previously used notation $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$ can indicate the configurations of Z_2, so the above matrix $\mathcal{B}$ have columns indexed by $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$. The definition of $m_i(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)$, $i = 1, 2, 3$ can be understood as certain 2×2 minor of the matrix $\mathcal{B}$. Denote the determinant of a matrix $\mathbf{C}$ by $|\mathbf{C}|$. In particular, we have the following equalities,

$$m_1(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,1,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(1,0,\boldsymbol{\alpha}_1^*)}p_{(0,1,\boldsymbol{\alpha}_2^*)} = \begin{vmatrix} p_{(0,1,\boldsymbol{\alpha}_1^*)} & p_{(0,1,\boldsymbol{\alpha}_2^*)} \\ p_{(1,0,\boldsymbol{\alpha}_1^*)} & p_{(1,0,\boldsymbol{\alpha}_2^*)} \end{vmatrix} = |\mathcal{B}(\{2,3\},\, \{\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*\})|,$$

$$m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(1,0,\boldsymbol{\alpha}_1^*)}p_{(0,0,\boldsymbol{\alpha}_2^*)} = \begin{vmatrix} p_{(0,0,\boldsymbol{\alpha}_1^*)} & p_{(0,0,\boldsymbol{\alpha}_2^*)} \\ p_{(1,0,\boldsymbol{\alpha}_1^*)} & p_{(1,0,\boldsymbol{\alpha}_2^*)} \end{vmatrix} = |\mathcal{B}(\{1,2\},\, \{\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*\})|,$$

$$m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = p_{(0,0,\boldsymbol{\alpha}_1^*)}p_{(0,1,\boldsymbol{\alpha}_2^*)} - p_{(0,1,\boldsymbol{\alpha}_1^*)}p_{(0,0,\boldsymbol{\alpha}_2^*)} = \begin{vmatrix} p_{(0,0,\boldsymbol{\alpha}_1^*)} & p_{(0,0,\boldsymbol{\alpha}_2^*)} \\ p_{(0,1,\boldsymbol{\alpha}_1^*)} & p_{(0,1,\boldsymbol{\alpha}_2^*)} \end{vmatrix} = |\mathcal{B}(\{1,3\},\, \{\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*\})|.$$

In the above display, the $\mathcal{B}(\{1,2\},\, \{\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*\})$ denotes the 2×2 submatrix of $\mathcal{B}$ containing the entries in rows with indices 1, 2 and columns $\boldsymbol{\alpha}_1^*$, $\boldsymbol{\alpha}_2^*$.

We can use the technical machinery in the last paragraph to discover some meaningful subsets of the non-identifiable null set $\mathcal{N}_{D,1} \cup \mathcal{N}_{D,2}$. First, define

$$\mathcal{N}_{1,\text{sub}} = \{m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = 0 \text{ for all } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-2}\}, \tag{S.23}$$

$$\mathcal{N}_{2,\text{sub}} = \{m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = 0 \text{ for all } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-2}\}. \tag{S.24}$$

According to the definition of $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$, it is clear that the two sets defined above satisfy $\mathcal{N}_{1,\text{sub}} \subseteq \mathcal{N}_{D,1}$ and $\mathcal{N}_{2,\text{sub}} \subseteq \mathcal{N}_{D,2}$. First consider the statistical implication of $\mathcal{N}_{1,\text{sub}}$. Since $m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = |\mathcal{B}(\{1,2\}, \{\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*\})|$, when $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*$ range over all the possible patterns in $\{0,1\}^{K-2}$, the $m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*)$ will take on values of all the possible 2×2 minors of the $2 \times 2^{(K-2)}$ matrix $\mathcal{B}(\{1,2\}, :)$ (i.e., the submatrix of $\mathcal{B}$ consisting of its first two rows). The assertion in $\mathcal{N}_{1,\text{sub}}$ that all these determinants equal zero essentially implies the submatrix $\mathcal{B}(\{1,2\}, :)$ has rank one, i.e., has the two rows proportional to each other. This means for all $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$, the ratio $p_{(1,0,\boldsymbol{\alpha}^*)}/p_{(0,0,\boldsymbol{\alpha}^*)}$ is a constant δ, which further implies the ratio $p_{(1,0,\boldsymbol{\alpha}^*)}/(p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)})$ is also a constant equal to $1/(1+1/\delta)$, which we denote by ρ:

$$\begin{aligned}\rho = \frac{p_{(1,0,\boldsymbol{\alpha}^*)}}{p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)}} &= \frac{\mathbb{P}(A_1 = 1, A_2 = 0, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)}{\mathbb{P}(A_2 = 0, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)} \\ &= \frac{\mathbb{P}(A_1 = 1, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0)}{\mathbb{P}(\mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0)}, \quad \forall \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}.\end{aligned}$$

So we have the following

$$\mathbb{P}(A_1 = 1, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0) = \rho \cdot \mathbb{P}(\mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0). \tag{S.25}$$

Now summing over the above equation for all $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$, we obtain

$$\begin{aligned}\sum_{\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}} \mathbb{P}(A_1 = 1, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0) = \rho \cdot \sum_{\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}} \mathbb{P}(\mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0), \\ \Longrightarrow \quad \mathbb{P}(A_1 = 1 \mid A_2 = 0) = \rho.\end{aligned}$$

Plugging back $\rho = \mathbb{P}(A_1 = 1 \mid A_2 = 0)$ into (S.25) gives the following for all $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$,

$$\mathbb{P}(A_1 = 1, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0) = \mathbb{P}(A_1 = 1 \mid A_2 = 0) \cdot \mathbb{P}(\mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0);$$

in a very similar fashion we can also obtain $\mathbb{P}(A_1 = 0, \mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0) = \mathbb{P}(A_1 = 0 \mid A_2 = 0) \cdot \mathbb{P}(\mathbf{A}_{3:K} = \boldsymbol{\alpha}^* \mid A_2 = 0)$ for all $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$. This essentially means attribute A_1 and attributes $\mathbf{A}_{3:K}$ are conditionally independent given $A_2 = 0$. So we have obtained that $\boldsymbol{p} \in \mathcal{N}_{1,\text{sub}}$ implies A_1 and $\mathbf{A}_{3:K}$ are conditionally independent given $A_2 = 0$. By symmetry, we similarly have that $\boldsymbol{p} \in \mathcal{N}_{2,\text{sub}}$ implies A_2 and $\mathbf{A}_{3:K}$ are conditionally independent given $A_1 = 0$. In summary, we have proved that $\mathcal{N}_{1,\text{sub}}$ and $\mathcal{N}_{2,\text{sub}}$ defined in (S.23)-(S.24) correspond to the following conditional independence statements,

$$\mathcal{N}_{1,\text{sub}} = \{\boldsymbol{p} \text{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_2 = 0)\} \subseteq \mathcal{N}_{D,1};$$
$$\mathcal{N}_{2,\text{sub}} = \{\boldsymbol{p} \text{ satisfies } (A_2 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_1 = 0)\} \subseteq \mathcal{N}_{D,2}.$$

Additionally, by the basic property of marginal independence and conditional independence, if $\boldsymbol{p}$ satisfies the marginal independence statement such as "$A_1 \perp\!\!\!\perp \mathbf{A}_{3:K}$", then it necessarily also satisfies the conditional independence statement "$A_1 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_2 = 0$". Therefore we have we also have

$$\mathcal{N}_{1,\text{sub}} = \{\boldsymbol{p} \text{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_2 = 0)\} \supseteq \{\boldsymbol{p} \text{ satisfies } (A_1 \perp\!\!\!\perp \mathbf{A}_{3:K})\};$$
$$\mathcal{N}_{2,\text{sub}} = \{\boldsymbol{p} \text{ satisfies } (A_2 \perp\!\!\!\perp \mathbf{A}_{3:K} \mid A_1 = 0)\} \supseteq \{\boldsymbol{p} \text{ satisfies } (A_2 \perp\!\!\!\perp \mathbf{A}_{3:K})\}.$$

Combining the two conclusions above, we have proved the first two conclusions in (12) in Proposition 3.

Next we prove the third conclusion in (12) in Proposition 3. Define

$$\mathcal{N}_{\text{both}} = \{m_2(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = m_3(\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^*) = 0 \text{ holds for all } \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-2}.\} \tag{S.26}$$

First note that $\mathcal{N}_{\text{both}} \subseteq \mathcal{N}_{D,1} \cup \mathcal{N}_{D,2}$ obviously holds according to definition of $\mathcal{N}_{D,1}$ and $\mathcal{N}_{D,2}$. We next examine the statistical implication the set $\mathcal{N}_{\text{both}}$. If $\boldsymbol{p} \in \mathcal{N}_{\text{both}}$, then we have the following for all $\boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-2}$,

$$p_{(0,0,\boldsymbol{\alpha}_1^*)} p_{(1,0,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)} p_{(1,0,\boldsymbol{\alpha}_1^*)} = p_{(0,0,\boldsymbol{\alpha}_1^*)} p_{(0,1,\boldsymbol{\alpha}_2^*)} - p_{(0,0,\boldsymbol{\alpha}_2^*)} p_{(0,1,\boldsymbol{\alpha}_1^*)} = 0;$$
$$\Longrightarrow \quad \frac{p_{(1,0,\boldsymbol{\alpha}_1^*)}}{p_{(0,0,\boldsymbol{\alpha}_1^*)}} = \frac{p_{(1,0,\boldsymbol{\alpha}_2^*)}}{p_{(0,0,\boldsymbol{\alpha}_2^*)}}, \quad \frac{p_{(0,1,\boldsymbol{\alpha}_1^*)}}{p_{(0,0,\boldsymbol{\alpha}_1^*)}} = \frac{p_{(0,1,\boldsymbol{\alpha}_2^*)}}{p_{(0,0,\boldsymbol{\alpha}_2^*)}}, \quad \forall \boldsymbol{\alpha}_1^*, \boldsymbol{\alpha}_2^* \in \{0,1\}^{K-2}.$$

This implies there exist some constants ρ_1, ρ_2 such that

$$\frac{p_{(1,0,\boldsymbol{\alpha}^*)}}{p_{(0,0,\boldsymbol{\alpha}^*)}} = \rho_1, \quad \frac{p_{(0,1,\boldsymbol{\alpha}^*)}}{p_{(0,0,\boldsymbol{\alpha}^*)}} = \rho_2, \quad \forall \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}. \tag{S.27}$$

Then for arbitrary $(x, y) \in \{(0,0),\ (0,1),\ (1,0)\}$ and $\boldsymbol{\alpha}^* \in \{0,1\}^{K-2}$, we will have

$$\begin{aligned}
&\mathbb{P}(\mathbf{A}_{1:2} = (x,y) \mid \mathbf{A}_{1:2} \neq (1,1),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*) \\
&= \frac{\mathbb{P}(\mathbf{A}_{1:2} = (x,y),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)}{\mathbb{P}(\mathbf{A}_{1:2} \neq (1,1),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)} \\
&= \frac{p_{(x,y,\boldsymbol{\alpha}^*)}}{p_{(0,0,\boldsymbol{\alpha}^*)} + p_{(0,1,\boldsymbol{\alpha}^*)} + p_{(1,0,\boldsymbol{\alpha}^*)}} = \frac{\frac{p_{(x,y,\boldsymbol{\alpha}^*)}}{p_{(0,1,\boldsymbol{\alpha}^*)}}}{1 + \frac{p_{(0,1,\boldsymbol{\alpha}^*)}}{p_{(0,1,\boldsymbol{\alpha}^*)}} + \frac{p_{(1,0,\boldsymbol{\alpha}^*)}}{p_{(0,1,\boldsymbol{\alpha}^*)}}} \\
&= \begin{cases} \dfrac{1}{1+\rho_1+\rho_2}, & \text{if } (x,y) = (0,0); \\ \dfrac{\rho_1}{1+\rho_1+\rho_2}, & \text{if } (x,y) = (1,0); \\ \dfrac{\rho_2}{1+\rho_1+\rho_2}, & \text{if } (x,y) = (0,1). \end{cases}
\end{aligned} \tag{S.28}$$

The above deduction implies that the conditional distribution $\mathbb{P}(\mathbf{A}_{1:2} = (x,y) \mid \mathbf{A}_{1:2} \neq (1,1),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)$ does not depend on $\mathbf{A}_{3:K}$ and hence can be indeed written as

$$\mathbb{P}(\mathbf{A}_{1:2} = (x,y) \mid \mathbf{A}_{1:2} \neq (1,1),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*) = \mathbb{P}(\mathbf{A}_{1:2} = (x,y) \mid \mathbf{A}_{1:2} \neq (1,1)).$$

Statistically, the above observation means the conditional independence $(\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K} \mid \mathbf{A}_{1:2} \neq (1,1))$ holds. Also, note that in order for $\mathbb{P}(\mathbf{A}_{1:2} = (x,y) \mid \mathbf{A}_{1:2} \neq (1,1),\ \mathbf{A}_{3:K} = \boldsymbol{\alpha}^*)$ in (S.28) to not depend on $\boldsymbol{\alpha}^*$, we must have (S.27) holds for some constants ρ_1, ρ_2. In summary, we have shown that $\boldsymbol{p} \in \mathcal{N}_{\text{both}}$ if and only if $(\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K} \mid \mathbf{A}_{1:2} \neq (1,1))$ holds. Namely, the $\mathcal{N}_{\text{both}}$ defined in (S.26) can be equivalently written as

$$\mathcal{N}_{\text{both}} = \{\boldsymbol{p} \text{ satisfies } (\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K} \mid \mathbf{A}_{1:2} \neq (1,1))\}.$$

Finally, recall that we have $\mathcal{N}_{\text{both}} \subseteq \mathcal{N}_{D,1} \cup \mathcal{N}_{D,2}$, so

$$\mathcal{N}_{D,1} \cup \mathcal{N}_{D,2} \supseteq \{\boldsymbol{p} \text{ satisfies } (\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K} \mid \mathbf{A}_{1:2} \neq (1,1))\} \supseteq \{\boldsymbol{p} \text{ satisfies } (\mathbf{A}_{1:2} \perp\!\!\!\perp \mathbf{A}_{3:K})\}.$$

This completes the proof of Proposition 3. $\square$